NEHRU COLLEGE OF ENGINEERING AND RESEARCH CENTRE
(NAAC Accredited)
(Approved by AICTE, Affiliated to APJ Abdul Kalam Technological University, Kerala)

DEPARTMENT OF COMPUTER SCIENCE AND ENGINEERING

COURSE MATERIALS

MAT 102 VECTOR CALCULUS, DIFFERENTIAL EQUATIONS AND TRANSFORMS

VISION OF THE INSTITUTION

To mould true citizens who are millennium leaders and catalysts of change through excellence in education.

MISSION OF THE INSTITUTION

NCERC is committed to transform itself into a center of excellence in Learning and Research in Engineering and Frontier Technology and to impart quality education to mould technically competent citizens with moral integrity, social commitment and ethical values.

We intend to facilitate our students to assimilate the latest technological know-how and to imbibe discipline, culture and spirituality, and to mould them in to technological giants, dedicated research scientists and intellectual leaders of the country who can spread the beams of light and happiness among the poor and the underprivileged.

ABOUT DEPARTMENT

- Established in: 2002
- Course offered : B.Tech in Computer Science and Engineering

  M.Tech in Computer Science and Engineering

  M.Tech in Cyber Security
- Approved by AICTE New Delhi and Accredited by NAAC
- Affiliated to the University of A P J Abdul Kalam Technological University.

DEPARTMENT VISION

Producing Highly Competent, Innovative and Ethical Computer Science and Engineering Professionals to facilitate continuous technological advancement.

DEPARTMENT MISSION

1. To Impart Quality Education by creative Teaching Learning Process
2. To Promote cutting-edge Research and Development Process to solve real world problems with emerging technologies.
3. To Inculcate Entrepreneurship Skills among Students.
4. To cultivate Moral and Ethical Values in their Profession.

PROGRAMME EDUCATIONAL OBJECTIVES

PEO1: Graduates will be able to Work and Contribute in the domains of Computer Science and Engineering through lifelong learning.

PEO2: Graduates will be able to Analyse, design and development of novel Software Packages, Web Services, System Tools and Components as per needs and specifications.

PEO3: Graduates will be able to demonstrate their ability to adapt to a rapidly changing environment by learning and applying new technologies.

PEO4: Graduates will be able to adopt ethical attitudes, exhibit effective communication skills, Teamworkand leadership qualities.

PROGRAM OUTCOMES (POS)

Engineering Graduates will be able to:

1. **Engineering knowledge**: Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization to the solution of complex engineering problems.
2. **Problem analysis**: Identify, formulate, review research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.
3. **Design/development of solutions**: Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.
4. **Conduct investigations of complex problems**: Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.
5. **Modern tool usage**: Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of the limitations.
6. **The engineer and society**: Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.
7. **Environment and sustainability**: Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.
8. **Ethics**: Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.
9. **Individual and team work**: Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.
10. **Communication**: Communicate effectively on complex engineering activities with the engineering community and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.
11. **Project management and finance**: Demonstrate knowledge and understanding of the engineering and management principles and apply these to one's own work, as a member and leader in a team, to manage projects and in multidisciplinary environments.
12. **Life-long learning**: Recognize the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

PROGRAM SPECIFIC OUTCOMES (PSO)

PSO1: Ability to Formulate and Simulate Innovative Ideas to provide software solutions for Real-time Problems and to investigate for its future scope.

PSO2: Ability to learn and apply various methodologies for facilitating development of high quality System Software Tools and Efficient Web Design Models with a focus on performance optimization.

PSO3: Ability to inculcate the Knowledge for developing Codes and integrating hardware/software products in the domains of Big Data Analytics, Web Applications and Mobile Apps to create innovative career path and for the socially relevant issues.

COURSE OUTCOMES

CO1	Compute the derivatives and line integrals of vector functions and learn their applications
CO2	Evaluate surface and volume integrals and learn their inter-relations and applications.
CO3	Solve homogeneous and non-homogeneous linear differential equation with constant coefficients.
CO4	Compute Laplace transform and apply them to solve ODEs arising in engineering
CO5	Determine the Fourier transforms of functions and apply them to solve problems arising in engineering.

MAPPING OF COURSE OUTCOMES WITH PROGRAM OUTCOMES

	PO 1	PO 2	PO 3	PO 4	PO 5	PO 6	PO 7	PO 8	PO 9	PO 10	PO 11	PO 12
CO1	3	3	3	3	2	1			1	2		2
CO2	3	3	3	3	2	1			1	2		2
CO3	3	3	3	3	2	1			1	2		2
CO4	3	3	3	3	2	1			1	2		2
CO5	3	3	3	3	2	1			1	2		2

Note: H-Highly correlated=3, M-Medium correlated=2, L-Less correlated=1

	PSO1	PSO2	PSO3
CO1			
CO2			
CO3			
CO4			
CO5			

SYLLABUS

MAT 102	VECTOR CALCULUS, DIFFERENTIAL EQUATIONS AND TRANSFORMS	CATEGORY	L	T	P	CREDIT	Year of Introduction
		BSC	3	1	0	4	2019

Preamble: This course introduces the concepts and applications of differentiation and integration of vector valued functions, differential equations, Laplace and Fourier Transforms. The objective of this course is to familiarize the prospective engineers with some advanced concepts and methods in Mathematics which include the Calculus of vector valued functions, ordinary differential equations and basic transforms such as Laplace and Fourier Transforms which are invaluable for any engineer's mathematical tool box. The topics treated in this course have applications in all branches of engineering.

Prerequisite: Calculus of single and multi variable functions.

Module 1 (Calculus of vector functions)

(Text 1: Relevant topics from sections 12.1, 12.2, 12.6, 13.6, 15.1, 15.2, 15.3)

Vector valued function of single variable, derivative of vector function and geometrical interpretation, motion along a curve-velocity, speed and acceleration. Concept of scalar and vector fields , Gradient and its properties, directional derivative , divergence and curl, Line integrals of vector fields, work as line integral, Conservative vector fields , independence of path and potential function(results without proof).

Module 2 (Vector integral theorems)

(Text 1: Relevant topics from sections 15.4, 15.5, 15.6, 15.7, 15.8)

Green's theorem (for simply connected domains, without proof) and applications to evaluating line integrals and finding areas. Surface integrals over surfaces of the form z = g(x, y), y = g(x, z) or x = g(y, z) , Flux integrals over surfaces of the form z = g(x, y), y = g(x, z) or x = g(y, z), divergence theorem (without proof) and its applications to finding flux integrals, Stokes' theorem (without proof) and its applications to finding line integrals of vector fields and work done.

Module- 3 (Ordinary differential equations)

(Text 2: Relevant topics from sections 2.1, 2.2, 2.5, 2.6, 2.7, 2.10, 3.1, 3.2, 3.3)

Homogenous linear differential equation of second order, superposition principle,general solution, homogenous linear ODEs with constant coefficients-general solution. Solution of Euler-Cauchy equations (second order only).Existence and uniqueness (without proof). Non homogenous linear ODEs-general solution, solution by the method of undetermined coefficients (for the right hand side of the form x^n,e^{kx},$sinax$,$cosax$,$e^{kx}sinax$$e^{kx}cosax$and their linear combinations), methods of variation of parameters. Solution of higher order equations-homogeneous and non-homogeneous with constant coefficient using method of undetermined coefficient.

Module- 4 (Laplace transforms)

(Text 2: Relevant topics from sections 6.1,6.2,6.3,6.4,6.5)

Laplace Transform and its inverse ,Existence theorem (without proof) , linearity,Laplace transform of basic functions, first shifting theorem, Laplace transform of derivatives and integrals, solution of differential equations using Laplace transform, Unit step function, Second shifting theorems. Dirac delta function and its Laplace transform, Solution of ordinary differential equation involving unit step function and Dirac delta functions. Convolution theorem(without proof)and its application to finding inverse Laplace transform of products of functions.

Module- 4 (Laplace transforms)

(Text 2: Relevant topics from sections 6.1,6.2,6.3,6.4,6.5)

Laplace Transform and its inverse ,Existence theorem (without proof) , linearity,Laplace transform of basic functions, first shifting theorem, Laplace transform of derivatives and integrals, solution of differential equations using Laplace transform, Unit step function, Second shifting theorems. Dirac delta function and its Laplace transform, Solution of ordinary differential equation involving unit step function and Dirac delta functions. Convolution theorem(without proof)and its application to finding inverse Laplace transform of products of functions.

Module-5 (Fourier Tranforms)

(Text 2: Relevant topics from sections 11.7,11.8, 11.9)

Fourier integral representation, Fourier sine and cosine integrals. Fourier sine and cosine transforms, inverse sine and cosine transform. Fourier transform and inverse Fourier transform, basic properties. The Fourier transform of derivatives. Convolution theorem (without proof)

Text Books

1. H. Anton, I. Biven S.Davis, "Calculus", Wiley, 10th edition, 2015.
2. Erwin Kreyszig, "Advanced Engineering Mathematics", Wiley, 10th edition, 2015.

QUESTION BANK

MODULE I

Q:NO:	QUESTIONS	CO	KL	PAGE NO:
1	Find the natural domain of $r(t) = sin2ti - 4tj$	CO1	K2	12
2	Find velocity, acceleration and speed if $r(t) = 3ti + 2t^2 j + tk$	CO1	K3	13
3	Find the directional derivative of f(x,y)=$e^x cosy$ at P(0,$\frac{\pi}{4}$) in the direction of $a = 5i - 2j$	CO1	K1	15
4	Find the divergence and curl of the vector field $f(x, y, z) = x^2 y\, i + 2y^3 z\, j + 3zk$	CO1	K2	18
5	Is the vector $\boldsymbol{r}$ where $\boldsymbol{r}$ = $x\boldsymbol{i}$ + $y\boldsymbol{j}$ + $z\boldsymbol{k}$ conservative. Justify your answer.	CO1	K1	16
6	Prove that the force field $\boldsymbol{F}$ = $e^y \boldsymbol{i}$ + $xe^y\, \boldsymbol{j}$ is conservative in the entire xy-plane	CO1	K4	17
7	Find the work done by the force field$\boldsymbol{F}(x, y, z)$ = $xy\boldsymbol{i}$ + $yz\boldsymbol{j}$ + $xz\boldsymbol{k}$ along C where C is the curve$\boldsymbol{r}(t)$ = $t\boldsymbol{i}$ + $t^2 \boldsymbol{j}$ + $t^3 \boldsymbol{k}$	CO1	K3	23
8	Find the divergence of the vector field F $= \frac{c}{(x^2+y^2+z^2)^{3/2}}$ (xi + yj + zk)	CO1	K4	28
9	Find the work done by the force field F = ($e^x - y^3$) + (cos $y + x^3$)j on a particle that travels once around the unit circle centred at origin having radius 1.	CO1	K3	24
10	How would you calculate the speed, velocity and acceleration at any instant of a particle moving in space whose position vector at time $t$is (t)?	CO1	K1	14
11	When do you say that a vector field is conservative? What are the implications if a vector field is conservative?	CO1	K4	17

MODULE II

1	Use the divergence theorem to find the outward flux of the vector field $F(x, y, z) = z\boldsymbol{k}$ across the x^2 i+ y^2j + z^2k = a^2	CO2	K2	35
2	State Greens theorem including all the required hypotheses	CO2	K1	32
3	What is the outward flux of $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ across any unit cube.	CO2	K3	37
4	What is the relationship between Green's theorem and Stokes theorem?	CO2	K2	46
5	Use divergence theorem to find the outward flux of the vector field $\boldsymbol{F} = 2x\boldsymbol{i} + 3y\boldsymbol{j} + z^3\ \boldsymbol{k}$ acrossthe unit cube bounded by or $x = 0$, $y = 0$,$z = 0$, $x = 1$,$y = 1$,$z = 1$	CO2	K4	36
6	Find the circulation of $\boldsymbol{F} = (x - z)\boldsymbol{i} + (y - x)\boldsymbol{j} + (\boldsymbol{z} - \boldsymbol{xy})\boldsymbol{k}$ using Stokes theorem around the triangle with vertices $A(1,0,0)$, $B(0,2,0)$and $C(0,0,1)$	CO2	K3	47
7	Use divergence theorem to find the volume of the cylindrical solid bounded by x^2 i+ $4x + y^2$ k= 7, $z = -1$, $z = 4$, given the vector field $\boldsymbol{F} = \boldsymbol{x}i + \boldsymbol{y}j + \boldsymbol{z}k$ across surfaceof the cylinder	CO2	K3	38
8	Use Stokes theorem to evaluate$\int \boldsymbol{F}.\boldsymbol{dr}$ where$\boldsymbol{F} = x^2$ ii + $3x\boldsymbol{j} - y^3\ \boldsymbol{k}$where Cis the circle $x^2 + y^2 = 1$ in the xy- plane with counterclockwise orientation looking down the positive z-axis	CO2	K2	46
9	Use Green's theorem to express the area of a plane region bounded by a curve as a line integral.	CO2	K2	33
10	Write any one application each of line integral, double integral and surface integral.	CO2	K1	50

MODULE III

1	Solve $y'' + 4y' + 2.5y = 0$	CO3	K2	52
2	Does the function $y = C_1 \cos x + C_2\, sinx$ form a solution of $y'' + y = 0$?. Is it the general solution? Justify your answer	CO3	K3	53
3	Solve the differential equation $y'' + y = 0.001x^2$ using method of undetermined coefficient.	CO3	K3	57
4	Solve the differential equation of $y''' - 3y'' + 3y' - y = e^x - x - 1$.	CO3	K3	58
5	Solve $y'' + 4y' + 4y = x^2 + e^{-x}\cos x$	CO3	K2	60
6	Solve $y''' + 3y'' + 3y' + y = 30\, e^{-x}$ given $y(0) = 3, y'(0) = -3$, $y''(0) = -47$	CO3	K3	65
7	Using method of variation of parameters, solve $y'' + y = sec\, x$	CO3	K3	72
8	Find the characteristic roots of the ODE $y'' + 2y' + 5y = 0$	CO3	K2	54
9	State Super position principle	CO3	K1	53
10	Solve by method of variation of parameters $y'' + 4y = \cos 2x$	CO3	K2	73
11	.Solve $x^2y'' + 0.7xy' - 0.1\, y = 0$	CO3	K2	54
12	Find an ODE for the given basis $e^{\sqrt{3}x}$, $xe^{\sqrt{3}x}$.	CO3	K2	55

MODULE IV

1	Find the Laplace inverse transform of $\frac{1}{s(s^2+w^2)}$	CO4	K1	78
2	State the Convolution theorem for Laplace transform	CO4	K1	80
3	Solve the differential equation $y'' - y = t \quad y(0) = 1, y'(0) = 1$ using Laplace transform	CO4	K2	86
4	Apply Convolution theorem to find the inverse Laplace transform of $\frac{1}{s^2(s^2+w^2)}$	CO4	K3	82
5	Using Laplace transform , solve the IVP $y'' - 3y' + 2y = 4e^{2t} \qquad y(0) = -3, y'(0) = 5$	CO4	K2	87
6	Find Laplace transform $te^{2t}\sin 3t$	CO4	K1	79
7	Find inverse Laplace transform of $\frac{2(e^{-s}-e^{-3s})}{(S^2-4)}$	CO4	K1	83
8	Solve $y'' + 3y' + 2y = f(t)\ where\ f(t) = 1\ for\ 0 < t < 1\ and\ f(t) = 0\ for\ t > 1$ using Laplace Transform	CO4	K3	91
9	Apply Convolution theorem to find the inverse Laplace transform of $\frac{1}{s^2(s^2+w^2)}$	CO4	K4	86
10	Using Laplace transform , solve the IVP $y'' + 4y' + 5y = \delta(t-1) \qquad y(0) = 0, y'(0) = 3$	CO4	K3	91

MODULE V

1	State the Convolution theorem for Fourier transform	CO5	K1	96
2	Find the Fourier transform of $f(x) = \|x\|\ if\ \|x\| < 1$ and f(x)=0 other wise	CO5	K2	97
3	Find the Fourier cosine and sine transform of $\begin{cases} k & if\ \ 0 < x < a \\ 0 & if\ \ \ \ \ \ x > a \end{cases}$	CO5	K3	98
4	Find the Fourier cosine transform of $e^{-ax} for\ a > 0$	CO5	K3	100
5	Find the Fourier transform of e^{-x^2} .Hence find $F(xe^{-x^2})$	CO5	K2	99
6	Using Fourier Integral representation show that $\int_0^\infty \frac{sinw - wcosw}{w^2}\ sinxwdw = \begin{cases} \frac{\pi}{2}x & , if\ 0 < x < 1 \\ \frac{\pi}{4} & , \text{if } x = 1 \\ 0 , & \text{if } x > 1 \end{cases}$	CO5	K4	95
7	Using Fourier Integral representation show that $\int_0^\infty \frac{cosx\omega + \omega sinx\omega}{1+w^2}\ dx = \begin{cases} 0 & , if\ x < 0 \\ \frac{\pi}{2} & , \text{if } x = 0 \\ \pi e^{-x} , & \text{if } x > 0 \end{cases}$	CO5	K4	97
8	Find the Fourier Sine and cosine transform of $\frac{e^{-ax}}{x}$	CO5	K2	101
9	Find the Fourier integral representation for $f(x) = e^{-kx}\ for\ x > 0\ and\ k > 0\ and\ hence\ evaluate$ $\int_0^\infty \frac{cos\omega x}{k^2+\omega^2} d\omega$	CO5	K5	96
10	Does the Fourier sine transform of $f(x) = x^{-1} sinx\ \ for\ 0 < x < \infty$ exist? Justify your answer.	CO5	K3	111

Module I

Calculus of vector functions

Vector valued function of a single variable

If $r(t)$ is a vector valued function in 3 space, then for each variable 't' the vector $r = r(t)$ can be represented in terms of components as $\boxed{r = r(t) = \langle x(t), y(t), z(t)\rangle = x(t)i + y(t)j + z(t)k}$

The functions $x(t), y(t), z(t)$ are called the component functions or the components of $r(t)$.

Eg: $r(t) = \langle t, t^2, t^3\rangle = ti + t^2 j + t^3 k$

then $x(t) = t$, $y(t) = t^2$, $z(t) = t^3$.

The domain of a vector valued function '$r(t)$' is the set of allowable values for t. If $r(t)$ is defined in terms of component functions and the domain is not specified explicitely then it will be understood that domain is the intersection of the natural domains of the component functions.

Pbms

1) Find the natural domain of $r(t) = \langle \ln|t-1|, e^t, \sqrt{t}\rangle$

Ans. $r(t) = \ln|t-1|\, i + e^t j + \sqrt{t}\, k$.

$x(t) = \ln|t-1| \to$ Domain, $(-\infty,1) \cup (1,\infty)$

$y(t) = e^t$ Domain $(-\infty,\infty)$

$z(t) = \sqrt{t}$ Domain $[0,\infty]$

Intersection $[0,1) \cup (1,\infty)$

Domain of $r(t) = [0,1) \cup (1,\infty)$

2) Find the domain of $r(t) = \sin 2t\, i - 4t\, j$.

A: $x(t) = \sin 2t \to$ Domain $(-\infty,\infty)$

$y(t) = -4t$, Domain $(-\infty,\infty)$

Intersection $(-\infty,\infty)$

$\therefore$ Domain of $r(t) = (-\infty,\infty)$

Derivatives

If $r(t)$ is a vector valued function, we define the derivatives of 'r' with respect to 't' to be the vector valued function r' given by $r'(t) = \lim_{h \to 0} \frac{r(t+h) - r(t)}{h}$.

The domain r' consists of all values of t in the domain of $r(t)$ for which the limit exists.

The derivatives of $r(t)$ can be expressed as

$$\frac{d}{dt}[r(t)], \quad \frac{dr}{dt}, \quad r'(t), \quad r'$$

Eg: Let $r(t) = t^2 i + e^t j - 2\cos \pi t \, k$ then

$$r'(t) = 2t i + e^t j + 2\pi \sin \pi t \, \bar{k}$$

Pbms

1) find $r'(t)$ if (1) $r(t) = 6i - \sin t \, j$

(2) $r(t) = \tan^{-1} t \, i + t\cos t \, j - 2\sqrt{t} \, \bar{k}$

Ans

1) $r(t) = 6i - \sin t \, j$

$r'(t) = -\cos t \, j$.

(2) $\tan^{-1} t \, i + t\cos t \, j - 2\sqrt{t} \, k$

$$r'(t) = \frac{1}{1+t^2} i + [t \times -\sin t + \cos t] j - 2\frac{1}{2\sqrt{t}} k$$

$$= \frac{1}{1+t^2} i + [\cos t - t\sin t] j - \frac{1}{\sqrt{t}} k$$

Rules of Differenciation

Let $r(t)$, $r_1(t)$, $r_2(t)$ be differentiable vector valued functions that are all in 2 space or all in 3 space and let $f(t)$ be a differentiable real valued function, K scalar and c is a constant vector. Then the following rules of differenciation holds.

(1) $\frac{d}{dt}(c) = 0.$

(2) $\frac{d}{dt}\left[k\,r(t)\right] = k\,\frac{d}{dt}\left[r(t)\right].$

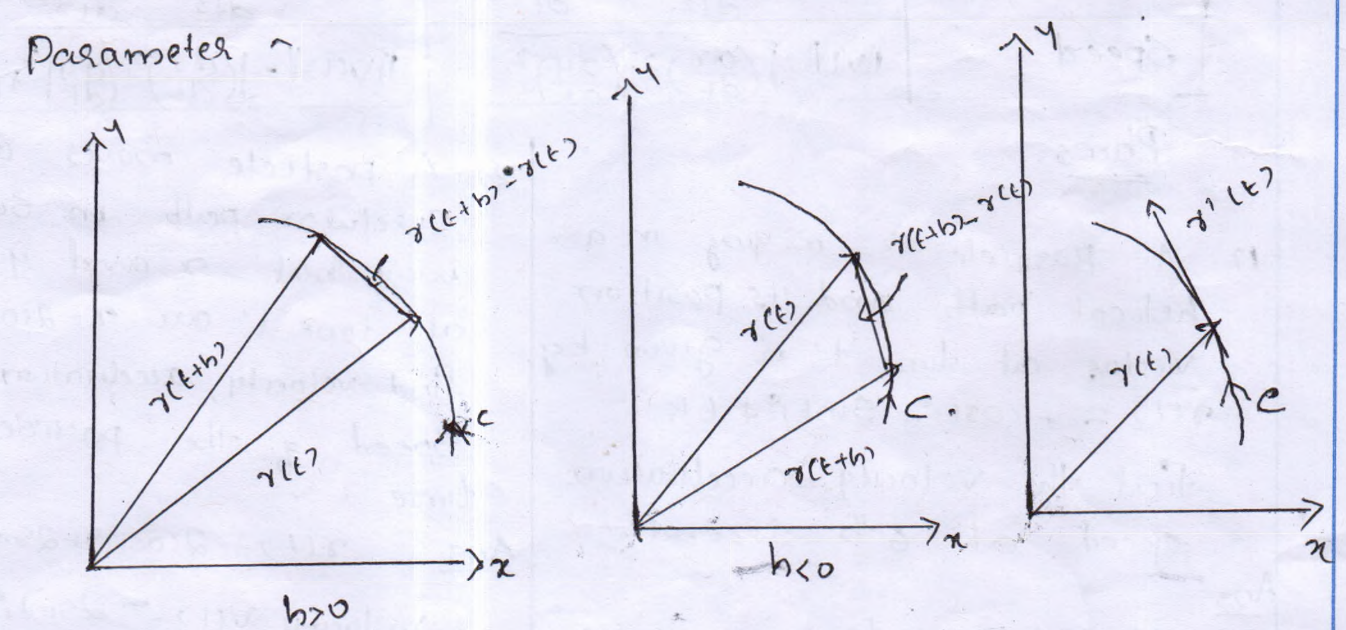
(3). $\frac{d}{dt}\left[r_1(t) + r_2(t)\right] = \frac{d}{dt} r_1(t) + \frac{d}{dt} r_2(t)$

(4) $\frac{d}{dt}\left[r_1(t) - r_2(t)\right] = \frac{d}{dt} r_1(t) - \frac{d}{dt} r_2(t)$

(5) $\frac{d}{dt}\left[f(t)\,r(t)\right] = f(t)\frac{d}{dt}\left[r(t)\right] + \frac{d}{dt}\left[f(t)\right]\cdot r(t)$

Geometric interpretation of Derivative.

Suppose that 'C' is the graph of a vector valued function $r(t)$ in 2-space or 3-space and that $r'(t)$ exists and is nonzero for a given value of t. If the vector $r'(t)$ is positioned with its initial point at the terminal point of the radius vector $r(t)$, then $r'(t)$ is tangent to C and points in the direction of increasing parameter.



If $r(t)$ is a vector valued function, then r is differentiable at 't' if and only if each of its component functions is differentiable at t, in which case the component fns of $r'(t)$ are the derivatives of the corresponding component fns of $r(t)$.

$$r'(t) = x'(t)\,i + y'(t)\,j.$$

Motion along a curve.

If $r(t)$ is the position vector of a particle moving along a curve in 2-space or 3-space, then the instantaneous velocity, instantaneous acceleration and instantaneous speed of the particle at time 't' are defined by.

$$\text{Velocity} = v(t) = \frac{dr}{dt}$$

$$\text{Acceleration} \quad a(t) = \frac{dv}{dt} = \frac{d^2r}{dt^2}$$

$$\text{Speed} = \|v(t)\| = \frac{ds}{dt}$$

	2 space	3 space
Position	$r(t) = x(t)\hat{i} + y(t)\hat{j}$	$r(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)k$
Velocity	$v(t) = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$	$v(t) = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}k$
Acceleration	$a(t) = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j}$	$a(t) = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}k$
Speed	$\|v(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$	$\|v(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$

Pbms

1) A particle is moving in a helical path and its position vector at time 't' is given by $r(t) = -\cos t\,\hat{i} + \sin t\,\hat{j} + tk$.

Find the velocity, acceleration speed at 6^{th} second.

Ans

Velocity $v(t) = \frac{dr}{dt} = \sin t\,\hat{i} + \cos t\,\hat{j} + k$

Velocity at $t = 6$ is $\sin 6\,\hat{i} + \cos 6\,\hat{j} + k$

Acceleration $a(t) = \frac{dv}{dt} = +\cos t\,\hat{i} - \sin t\,\hat{j}$

at $t = 6$ $a(t) = \cos 6\,\hat{i} - \sin 6\,\hat{j}$

Speed $= \|v(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

2) A particle moves along a circular path in such a way that x and y co-ordinates at time 't' are $x = 2\cos t$, $y = 2\sin t$. Find velocity, acceleration and speed of the particle at time t.

Ans: $r(t) = 2\cos t\,\hat{i} + 2\sin t\,\hat{j}$.

Velocity $v(t) = -2\sin t\,\hat{i} + 2\cos t\,\hat{j}$

Acceleration $a(t) = -2\cos t\,\hat{i} - 2\sin t\,\hat{j}$

Speed $= \|v(t)\|$

$= \sqrt{4\sin^2 t + 4\cos^2 t}$

$= 2$.

3) A Particle moves through 3-Space in such a way that its velocity is $V(t) = i + tj + t^2k$. Find the co-ordinates of the particle at time $t=1$ given that the particle is at the point $(-1,2,4)$ at time $t=0$.

→ Given $v(t) = i + tj + t^2k$.

$$r(t) = \int v(t)\,dt = \int [i + tj + t^2k]\,dt$$

$$= ti + \frac{t^2}{2}j + \frac{t^3}{3}k + C \longrightarrow (1).$$

given $t=0$. $r = -i + 2j + 4k$

$\Longrightarrow -i + 2j + 4k = C$. (Sub in (1).)

$$r(t) = ti + \frac{t^2}{2}j + \frac{t^3}{3}k - i + 2j + 4k$$

$$= (t-1)i + \left(\frac{t^2}{2} + 2\right)j + \left(\frac{t^3}{3} + 4\right)k$$

at $t=1$ $r(1) = (1-1)i + (1/2 + 2)j + (1/3 + 4)k = 0i + \frac{5}{2}j + \frac{13}{3}k$

∴ The co-ordinate of the partice at $t=1$ is $(0, 5/2, 13/3)$

4) $r(t) = 3ti + 2t^2j + tk$

Find velocity, Acceleration & speed at 3rd second

→ Velocity $V(t) = 3i + 4tj + k$. at $t=3$ $V = 3i + 12j + k$

Acceleration $a(t) = 4j$ at $t=3$ $a = 4j$

Speed $= \|v(t)\| = \sqrt{3^2 + 16t^2 + 1} = \sqrt{10 + 16t^2}$

at $t=3$ $= \sqrt{10 + 16(3)^2} = \sqrt{154}$

Gradient:

If f is a function of x and y then gradient of f is defined by $\nabla f(x,y) = f_x(x,y)i + f_y(x,y)j$

If f is a function of 3 variables x, y and z then gradient of f is defined by

$$\nabla f(x,y,z) = f_x(x,y,z)\,i + f_y(x,y,z)\,j + f_z(x,y,z)\,k$$

Del Operator $\quad \nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$.

[vector Differential Operator]

Properties of Gradient

- If $\phi(x,y,z)$ is a constant function then $\nabla\phi = 0$ fun
- $\nabla(\alpha\phi) = \alpha\nabla\phi \quad (\alpha \text{ Scalar})$
- $\nabla(\phi \pm \psi) = \nabla\phi \pm \nabla\psi$
- $\nabla(\phi.\psi) = \phi\nabla\psi + \psi\nabla\phi$.

Pbm Find ∇z where $z = \dfrac{6xe^{3y}}{x+8y}$

$\rightarrow \quad \nabla z = f_x\, i + f_y\, j$

$$f_x = \frac{(x+8y)\,6e^{3y} - 6xe^{3y}}{(x+8y)^2} = \frac{48ye^{3y}}{(x+8y)^2}.$$

$$f_y = \frac{(x+8y)\,18xe^{3y} - 6xe^{3y} \times 8}{(x+8y)^2} = \frac{6xe^{3y}[3+24y-8]}{(x+3y)^2}.$$

$$\therefore \nabla z = \frac{48ye^{3y}}{(x+8y)^2}\, i + \frac{6xe^{3y}[3+24y-8]}{(x+8y)^2}\, j$$

Applications of Gradient

In medical applications the operation of certain diagnostic equipment is designed to locate heat sources generated by tumors or infections, and in military applications the trajectories of heat seeking missiles are controlled to seek and destroy enemy aircraft.

Directional derivatives

If $f(x,y)$ is differentiable at (x_0,y_0) and if $u = u_1 i + u_2 j$ is a unit vector then the directional derivative $D_u f(x_0,y_0)$ in the direction of u is given by

$$\boxed{\begin{gathered} D_u f(x_0,y_0) = f_x(x_0,y_0)\,u_1 + f_y(x_0,y_0)\,u_2 \\ \text{or} \\ D_u f = \nabla f \cdot u \end{gathered}}$$

If $f(x,y,z)$ is differentiable at (x_0,y_0,z_0) and $u = u_1 i + u_2 j + u_3 k$ is a unit vector. then $D_u f(x_0,y_0,z_0)$ in the direction of $\bar{u}$ is

$$\boxed{\begin{gathered} D_u f(x_0,y_0,z_0) = f_x(x_0,y_0,z_0)\,u_1 + f_y(x_0,y_0,z_0)\,u_2 + f_z(x_0,y_0,z_0)\,u_3 \\ \text{or} \quad D_u f = \nabla f \cdot u \end{gathered}}$$

Pbms

1) Find $D_u f$ at P, $f(x,y) = \sin(5x-3y)$, $P(3,5)$, $u = \frac{3}{5} i - \frac{4}{5} j$.

$D_u f = f_x u_1 + f_y u_2 + f_z u_3$.

$f_x = \cos(5x-3y) \times 5$ $\quad f_x(3,5) = 5\cos(15-15) = 5$

$f_y = \cos(5x-3y) \times -3$ $\quad f_y(3,5) = -3\cos(15-15) = -3$

$u_1 = 3/5 \quad u_2 = -4/5$

$D_u = f_x u_1 + f_y u_2 = 5 \times 3/5 + -3 \times -4/5 = 3 + \frac{12}{5} = \frac{27}{5}$

2) Find the directional derivative of f at p in the direction of $\bar{a}$:

i) $f(x,y) = e^x \cos y$ $\quad P(0,\pi/4)$ $\quad a = 5i - 2j$

$\rightarrow$ $f_x = e^x \cos y$ $\quad f_x(0,\pi/4) = 1/\sqrt{2}$

$f_y = -e^x \sin y$ $\quad f_y(0,\pi/4) = -1/\sqrt{2}$

$u = \frac{a}{\|a\|} = \frac{5i - 2j}{\sqrt{5^2 + (-2)^2}} = \underbrace{\frac{5}{\sqrt{29}}}_{u_1} i - \underbrace{\frac{2}{\sqrt{29}}}_{u_2} j$

$D_u = f_x u_1 + f_y u_2 = \frac{1}{\sqrt{2}} \times \frac{5}{\sqrt{29}} + \frac{-1}{\sqrt{2}} \times \frac{-2}{\sqrt{29}}$

$= \frac{5+2}{\sqrt{58}} = \frac{7}{\sqrt{58}}$

2) $f(x,y,z) = \frac{z-x}{z+y}$ $\quad P(1,0,-3)$ $\quad a = -6i + 3j - 2k$

$\rightarrow$ $u = a/\|a\| = \frac{-6i + 3j + 2k}{\sqrt{(-6)^2 + 3^2 + (-2)^2}} = \frac{-6}{7} i + \frac{3}{7} j - \frac{2}{7} k$.

$f_x = -\frac{1}{z+y}$ $\quad f_x(1,0,-3) = 1/3$

$f_y = (z-x) \times -1/(z+y)^2$ $\quad f_y(1,0,-3) = \frac{4}{9}$

$f_z = \frac{(z+y) - (z-x)}{(z+4)^2} = \frac{4+x}{(z+4)^2}$ $\quad f_z(1,0,-3) = 1/9$

$D_u = f_x u_1 + f_y u_2 + f_z u_3 = 1/3 \times -6/7 + 4/9 \times 3/7 + -2/7 \times 1/9$

$= -8/63$

3) Find the directional derivative of $f = \frac{x-y}{x+y}$ at $(-1,-2)$ in the direction of a vector making a counter clockwise angle $\theta = \pi/2$ with the positive x-axis

$\rightarrow$ $f_x = \frac{(x+y) - (x-4)}{(x+y)^2} = \frac{2y}{(x+y)^2}$ $\quad f_x(-1,-2) = -4/9$.

$$f_y = \frac{(x+y)\times 1 - (x-4)}{(x+y)^2} = \frac{-2x}{(x+y)^2} \qquad f_y(-1,-2) = 2/9.$$

Unit vector u that makes an angle $\theta = \pi/2$ with the +ve x-axis is $u = \cos(\pi/2)\,i + \sin\pi/2\, j = j$.

$$D_u = -4/9 \times 0 + 2/9 \times 1 = \underline{2/9}$$

Let f be a function of 2 variables or 3 variables and let P denote the pt $P(x_0, y_0)$ or $P(x_0, y_0, z_0)$. Assume f is differentiable at P.

1. If $\nabla f = 0$ then all directional derivatives of f at P are zero

2. If $\nabla f \neq 0$ at P, then among all possible directional derivatives of f at P, the derevative in the direction of ∇f at P has the largest value. Value of this largest directional derivative is $\|\nabla f\|$ at P

3) If $\nabla f \neq 0$ at P, then among all possible directional derevatives of f at P, the derevative in the direction opposite to that of ∇f at P has the smallest value. The value of this smallest directional derevative is $-\|\nabla f\|$ at P

1 Let $f(x,y) = x^2 e^y$. Find the maximum value of directional derivative at $(-2,0)$ and find the unit vector in the direction in which maximum value occurs

$\rightarrow$ $\nabla f = f_x i + f_y j = 2xe^y i + x^2 e^y j$.

at $(-2,0)$ Gradient is

$$\nabla f = -4i + 4j$$

maximum value of directional derivatives $= \|\nabla f\|$
$= \sqrt{32} = 4\sqrt{2}$

maximum value occur in the direction of $\nabla f(-2,0)$

unit vector in this direction $= \frac{-4}{4\sqrt{2}} i + \frac{4}{4\sqrt{2}} j$

$= -1/\sqrt{2}\, i + 1/\sqrt{2}\, j$

Divergence and Curl.

If $F(x,y,z) = f(x,y,z)i + g(x,y,z)j + h(x,y,z)k$.

then we define the divergence of F, written.

$$\boxed{\operatorname{div} F = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \quad \text{or} \quad \operatorname{div} F = \nabla \cdot F}$$

$\operatorname{div} F = 0$. then F is called **Solenoidal** vector.

Curl of F written

$$\boxed{\operatorname{Curl} F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}}$$

$\operatorname{Curl} F = \nabla \times F = 0$ then F is called

Irrotational Vector or Conservative Vector field

Pbms

1 Find the divergence and curl of the vector field $F(x,y,z) = x^2y\,i + 2y^3z\,j + 3z\,k$.

$\longrightarrow$ $\operatorname{div} F = \nabla \cdot F = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(2y^3z) + \frac{\partial}{\partial z}(3z)$

$= 2xy + 6y^2z + 3.$

$\operatorname{curl} F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & 2y^3z & 3z \end{vmatrix}$

$= i\left(\frac{\partial}{\partial y}(3z) - \frac{\partial}{\partial z}(2y^3z)\right) - j\left[\frac{\partial}{\partial x}(3z) - \frac{\partial}{\partial z}(x^2y)\right]$
$+ k\left(\frac{\partial}{\partial x}(2y^3z) - \frac{\partial}{\partial y}(x^2y)\right)$

$= -2y^3 i - x^2 k$

---

2. S.T the divergence of the inverse square field $F(x,y,z) = \frac{c}{(x^2+y^2+z^2)^{3/2}}[xi + yj + zk]$ is zero.

$\longrightarrow$ $F = \frac{cx}{r^3}i + \frac{cy}{r^3}j + \frac{cz}{r^3}k$

$\because r = xi + yj + zk$, $r^2 = x^2+y^2+z^2$, $(x^2+y^2+z^2)^{3/2} = (r^2)^{3/2} = r^3$

$\operatorname{div} F = \nabla \cdot F = \frac{\partial}{\partial x}\left(\frac{cx}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{cy}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{cz}{r^3}\right)$

$\boxed{\frac{\partial r}{\partial x} = x/r \quad \frac{\partial r}{\partial y} = y/r \quad \frac{\partial r}{\partial z} = z/r}$

$\frac{\partial}{\partial x}\left(\frac{cx}{r^3}\right) = c\left[\frac{r^3 \cdot 1 - x \times 3r^2 \times \frac{\partial r}{\partial x}}{(r^3)^2}\right]$

$= c\left[\frac{r^3 - x \times 3r^2 \times x/r}{r^6}\right] = c\left[\frac{1}{r^3} - \frac{3x^2}{r^5}\right]$

|||ly $\frac{\partial}{\partial y}\left(\frac{cy}{r^3}\right) = c\left[\frac{1}{r^3} - \frac{3y^2}{r^5}\right]$ & $\frac{\partial}{\partial z}\left[\frac{cz}{r^3}\right] = c\left[\frac{1}{r^3} - \frac{3z^2}{r^5}\right]$

$\therefore \operatorname{div} F = c\left[\frac{1}{r^3} - \frac{x^2}{r^5} + 1/r^3 - \frac{y^2}{r^5} + 1/r^3 - \frac{z^2}{r^5}\right] = c\left[\frac{3}{r^3} - \frac{3(x^2+y^2+z^2)}{r^5}\right]$

$= c\left[\frac{3}{r^3} - \frac{3r^2}{r^5}\right] = c\left(\frac{3}{r^3} - 3/r^3\right) = 0$

3) Find div F & curl F $F(x,y,z) = e^{xy} i - 2\cos y\, j + \sin^2 z\, k$

$\rightarrow$ $\operatorname{div} F = \nabla \cdot F = \frac{\partial}{\partial x}(e^{xy}) + \frac{\partial}{\partial y}(-2\cos y) + \frac{\partial}{\partial z}(\sin^2 z)$

$= y e^{xy} + 2\sin y + 2\sin z \cos z$

$$\operatorname{curl} F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{xy} & -2\cos y & \sin^2 z \end{vmatrix} = i(0) - j(0) + k(0 - x e^{xy})$$

$= -x e^{xy} k$

4) $\bar{r} = xi + yj + zk$ S.T $\nabla\left(\frac{1}{\|r\|}\right) = -\frac{\bar{r}}{\|r\|^3}$.

$\rightarrow$ $r^2 = \|r\|^2 = x^2 + y^2 + z^2)$ let $r = \|r\|$.

We know that $\frac{\partial r}{\partial x} = x/r$ $\frac{\partial r}{\partial y} = y/r$ $\frac{\partial r}{\partial z} = z/r$

$\nabla(1/r) = i \frac{\partial}{\partial x}(1/r) + j\, \partial/\partial y\, (1/r) + k \frac{\partial}{\partial z}(1/r)$

$= i \times -1/r^2 \times \frac{\partial r}{\partial x} + j \times -1/r^2 \frac{\partial r}{\partial y} + k \times -1/r^2 \frac{\partial r}{\partial z}$.

$= -\frac{i}{r^2} \times x/r - j/r^2 \times y/r - \frac{k}{r^2} \times z/r$

$= -\frac{1}{r^3}[xi + yj + zk] = \frac{-\bar{r}}{r^3}$

5) Use chain rule S.T $\nabla f(r) = \frac{f'(r)}{r}(\bar{r})$ when $\bar{r} = xi + yj + zk$ $r = \|r\|$

$\rightarrow$ $\nabla f(r) = i \frac{\partial}{\partial x} f(r) + j \frac{\partial}{\partial y} f(r) + k \frac{\partial}{\partial z} f(r)$

$= i \cdot f'(r) \cdot \frac{\partial r}{\partial x} + j\, f'(r) \frac{\partial r}{\partial y} + k f'(r) \frac{\partial r}{\partial z}$

$= i \cdot f'(r) \times x/r + j\, f'(r)\, y/r + k f'(r) \cdot z/r$

$= \frac{f'(r)}{r}[xi + yj + zk]$

$= \frac{f'(r)}{r}\, \bar{r}$

Conservative fields and Potential functions

A vector field F in 2-space or 3-space is said to be <u>conservative</u> in a region if it is the gradient field for some function ϕ in the region, ie $F = \nabla\phi$, the function ϕ is called potential function for F in the region.

Probs

1) The function $\phi(x,y,z) = xy + yz + xz$ is potential for the vector field F. Find the vector field F.

$\rightarrow$ ϕ Potential $\Rightarrow$ $F = \nabla\phi$

$$= i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}.$$

$$i(y+z) + j(x+z) + k(y+x)$$

---

2) Confirm ϕ is a potential function for F where $\phi(x,y) = 2y^2 + 3x^2y - xy^4$ & $F(x,y) = (6xy - y^4)i + (4y + 3x^2 - 4xy^3)j$

$\longrightarrow$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$= i(6xy - y^4) + j\left[4y + 3x^2 - 4xy^3\right] = F.$$

---

3) Determine a so that $(x+3y)i + (y-2z)j + (x+az)k$ is solenoidal.

$\rightarrow$ F is solenoidal $\Rightarrow$ $\operatorname{div} F = 0$, $\nabla\cdot F = 0$

$$\nabla\cdot F = \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+az) = 0$$

$$= 1 + 1 + a = 0 \qquad 2 + a = 0 \qquad a = -2$$

<u>Conservative Vector field</u> $\nabla\times F = 0$.

or $\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial x}$. //.

1) Determine whether $F = 2e^{2x}\cos 2y\, i - 2e^{2x}\sin 2y\, j$ is a conservative vector field. If so find the potential function for it

$\longrightarrow$ F is conservative if $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$.

here $f = 2e^{2x}\cos 2y$ $g = -2e^{2x}\sin 2y$

$\frac{\partial f}{\partial y} = -4e^{2x}\sin 2y$

$\frac{\partial g}{\partial x} = -4e^{2x}\sin 2y$

$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \Longrightarrow F$ is conservative

$\therefore F = \nabla\phi \Longrightarrow 2e^{2x}\cos 2y\, i - 2e^{2x}\sin 2y\, j = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y}$

$\Rightarrow \frac{\partial\phi}{\partial x} = 2e^{2x}\cos 2y$ $\int$ w.r.to x $\phi = e^{2x}\cos 2y + c_1$

$\frac{\partial\phi}{\partial y} = -2e^{2x}\sin 2y$ $\int$ w.r.to y $\phi = e^{2x}\cos 2y + c_2$

$\therefore \phi = e^{2x}\cos 2y + c$

Conservative Vector field (3 variables).

$F = f(x,y,z)\, i + g(x,y,z)\, j + h(x,y,z)\, k$, is Conservative

This $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$, $\frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}$, $\frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}$

---

2) Determine whether $F = y\sin x\, i - \cos x\, j$ is Conservative. find potential function

$\longrightarrow$ $f = y\sin x$ $g = -\cos x$. $\frac{\partial f}{\partial y} = \sin x$ $\frac{\partial g}{\partial x} = \sin x$.

$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \Longrightarrow F$ is Conservative $\therefore F = \nabla\phi$

$y\sin x\, i - \cos x\, j = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y}$

$\frac{\partial\phi}{\partial x} = y\sin x \Longrightarrow \phi = -y\cos x + c_1$

$\frac{\partial\phi}{\partial y} = -\cos x \Longrightarrow \phi = -y\cos x + c_2$ $\Big\}$ $\phi = -y\cos x + c$

Line integrals

If C is a smooth curve parametrized by

$r(t) = x(t)i + y(t)j \quad (a \le t \le b)$.

$$\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t))\,\|r'(t)\|\,dt$$

|||ly $\quad r(t) = x(t)i + y(t)j + z(t)k \quad (a \le t \le b)$

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t))\,\|r'(t)\|\,dt$$

Pbms

1) Using given parametrization evaluate the line integral

$\int_C (1+xy^2)\,ds$

(a) $C: r(t) = ti + 2tj \quad 0 \le t \le 1$

(b) $C: r(t) = (1-t)i + (2-2t)j \quad 0 \le t \le 1$

→ (a) Given $r(t) = ti + 2tj \qquad x(t) = t, \quad y(t) = 2t$

$r'(t) = i + 2j \qquad \|r'(t)\| = \sqrt{5}$

$$\int_C (1+xy^2)\,ds = \int_0^1 \left(1 + t(2t)^2\right)\sqrt{5}\,dt = \sqrt{5}\int_0^1 1 + 4t^3\,dt$$

$$\sqrt{5}\left(t + \frac{4t^4}{4}\right)_0^1 = \underline{\underline{2\sqrt{5}}}$$

(b) $r(t) = (1-t)i + (2-2t)j \qquad x(t) = 1-t$

$\qquad y(t) = 2(1-t)$

$r'(t) = -i - 2j \qquad \|r'(t)\| = \underline{\underline{\sqrt{5}}}$

$$\int_C (1+xy^2)\,ds = \int_0^1 \left(1 + (1-t)(2(1-t))^2\right)\sqrt{5}\,dt$$

$$= \sqrt{5}\int_0^1 1 + 4(1-t)^3\,dt$$

$$= \sqrt{5}\int_0^1 1 + 4\left[1 - 3t + 3t^2 - t^3\right]dt$$

$$= \sqrt{5}\left[t + 4t - \frac{12t^2}{2} + \frac{12t^3}{3} - \frac{4t^4}{4}\right]_0^1 = \underline{\underline{2\sqrt{5}}}$$

2) ~~Consider~~ Evaluate the line integral $\int 3xyz\, ds$ where the curve C has parametrization $x=t$, $y=t^2$, $z=\frac{2}{3}t^3$ $0\le t\le 1$

$\longrightarrow$ $r(t) = t\hat{i} + t^2\hat{j} + \frac{2}{3}t^3$ $\quad r'(t) = \hat{i} + 2t\hat{j} + 2t^2\hat{k}$

$$\|r'(t)\| = \sqrt{1+4t^2+4t^4} = \sqrt{(1+2t^2)^2} = \underline{\underline{1+2t^2}}.$$

$$\int 3xyz\, ds = \int_0^1 3\cdot t \times t^2 \times \frac{2}{3}t^3 \times (1+2t^2)\, dt$$

$$= \int_0^1 2t^6(1+2t^2)\, dt = \int_0^1 2t^6 + 4t^8\, dt$$

$$\left[2\frac{t^7}{7} + 4\frac{t^9}{9}\right]_0^1 = \frac{2}{7} + 4/9 = \underline{\underline{\frac{46}{63}}}.$$

3) Evaluate the line integral $\int (xy+z^3)\, ds$ from $(1,0,0)$ to $(-1,0,\pi)$ along the helix 'c' that is represented by the parametric equations $x=\cos t$ $y=\sin t$ $z=t$ $(0\le t\le \pi)$

$\longrightarrow$ $r(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k}$ $\quad \|r'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} = \underline{\sqrt{2}}$

$r'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j} + \hat{k}$

$$\int f(x,y,z)\, ds = \int_0^\pi \left(\cos t \sin t + t^3\right] \sqrt{2}\, dt$$

$$= \int_0^\pi \left[\frac{\sin 2t}{2} + t^3\right) \sqrt{2}\, dt$$

$$= \sqrt{2}\left[-\frac{\cos 2t}{4} + \frac{t^4}{4}\right]_0^\pi$$

$$= \sqrt{2}\left[-\frac{1}{4} + \frac{\pi^4}{4} - \left(-\frac{1}{4}\right)\right) = \underline{\underline{\frac{\sqrt{2}\,\pi^4}{4}}}.$$

<u>Line integral of vector valued functions</u>

Let $x = x(t)$, $y = y(t)$, $z = z(t)$ $\quad a\le t\le b$ be the parametric eqns for C in which the orientation of C is in the direction of increasing t, Probs

1 Evaluate $\int_C 3xy\, dy$, where C is a line segment joining (0,0) and (1,2) with the given orientation.

a) [Oriented from (0,0) to (1,2)] C: $t\hat{i} + 2t\hat{j}$

b) [Oriented from (1,2) to (0,0)] C: $(1-t)\hat{i} + (2-2t)\hat{j}$

→ a) C: $t\hat{i} + 2t\hat{j}$ $\quad x = t \quad y = 2t$

$dy = 2dt$

$$\int_0^1 3 \times t \times 2t \times 2dt = 12\int_0^1 t^2 dt = 12\left[\frac{t^3}{3}\right]_0^1 = \underline{\underline{4}}$$

b) $x = 1-t \quad y = 2-2t \quad dy = -2dt$

$$\int_1^0 3(1-t)(2-2t) \times -2\, dt = -12\int_1^0 (1-t)^2 dt = \underline{+4}.$$

2) Evaluate $\int_C 2xy\, dx + (x^2+y^2)\, dy$ along the circular arc C given by $x = \cos t$, $y = \sin t$ $(0 \le t \le \pi/2)$

→ $x = \cos t \quad dx = -\sin t\, dt$

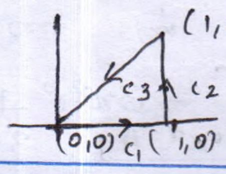
$y = \sin t \quad dy = \cos t\, dt$

$$\int_0^{\pi/2} 2\cos t \sin t \times -\sin t\, dt + [\cos^2 t + \sin^2 t] \times \cos t\, dt$$

$$\int_0^{\pi/2} -2\sin^2 t \cos t\, dt + \cos t\, dt$$

$$= -\frac{2\sin^3 t}{3}\Big]_0^{\pi/2} + \sin t\Big]_0^{\pi/2} = -2/3 + 1 = \underline{\underline{+1/3}}.$$

3). Evaluate $\int x^2 y\, dx + x\, dy$ where C is a triangular path shown in figure.



$$\longrightarrow \int_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$$

Along C_1 : $y=0$ $dy=0$ $x \to 0$ to 1

$$\int_{C_1} = \int_0^1 x^2 \times 0\, dx + x \times 0 = \underline{\underline{0}}$$

Along C_2 $(1,0)$ to $(1,2)$ $\frac{y-0}{2-0} = \frac{x-1}{0}$

$\Rightarrow x-1=0$ $\underline{\underline{x=1}}$

$y \to 0$ to 2 $\underline{\underline{dx=0}}$.

$$\int_{C_2} = \int_0^2 x^2 y \times 0 + x\, dy = \int_0^2 dy = [y]_0^2 = \underline{\underline{2}}$$

Along C_3 $(1,2)$ to $(0,0)$ $\frac{y-2}{0-2} = \frac{x-1}{0-1} \Rightarrow y-2=2(x-1)$

$\underline{\underline{y=2x}}$

$$\int_{C_3} = \int_1^0 x^2 \times 2x\, dx + x \times 2\, dx$$

$dy = 2dx$

$x \to 1$ to 0

$$\int_1^0 2x^3\, dx + 2x\, dx = \left[\frac{2x^4}{4} + \frac{2x^2}{2}\right]_1^0$$

$$= \underline{\underline{-3/2}}$$

$$\int_C f = \int_{C_1} + \int_{C_2} + \int_{C_3} = 0 + 2 - 3/2 = \underline{\underline{\frac{1}{2}}}.$$

Work Integral [Work done]

Work done $= \int_C F \cdot dr$, F is a continuous vector field & C is smooth oriented curve.

Line integral $\longrightarrow \int_C f \cdot dr$

Pbms

1) Find the workdone in moving a particle in the force field $F = 3x^2 i + (2xz - y)j + zk$ along the curve oriented by $x^2 = 4y$, $3x^3 = 8z$ from $x=0$ to $x=2$.

$\rightarrow$ Workdone $= \int \vec{f} \cdot d\vec{r}$

$$r = x\hat{i} + y\hat{j} + z\hat{k}$$

$$dr = dx\hat{i} + dy\hat{j} + dz\hat{k}.$$

$$\int f \cdot dr = \int \left[3x^2\hat{i} + (2xz - y)\hat{j} + 3\hat{k}\right] \cdot \left[dx\hat{i} + dy\hat{j} + dz\hat{k}\right]$$

$$= \int 3x^2 dx + (2xz - y)\, dy + z\, dz.$$

Here $x^2 = 4y \Rightarrow y = \frac{x^2}{4}$, $dy = \frac{x}{2} dx.$

$3x^3 = 8z \Rightarrow z = \frac{3}{8} x^3 \quad dz = 9/8\, x^2 dx$

$$\Rightarrow \int f \cdot dr = \int_0^2 3x^2 dx + \left\{2x \times \frac{3}{8} x^3 - \frac{x^2}{4}\right\} \frac{x}{2} dx + \frac{3x^3}{8}\, 9/8\, x^2 dx.$$

$$= \int_0^2 3x^2 + \frac{3x^5}{8} - \frac{x^3}{8} + \frac{27x^5}{64}\, dx$$

$$\left[\frac{3x^3}{3} + \frac{3 \cdot x^6}{8 \cdot 6} - \frac{x^4}{8 \cdot 4} + \frac{27\, x^6}{64 \times 6}\right]_0^2$$

$$= 8 + \frac{64}{16} - \frac{16}{32} + \frac{27 \times 64}{64 \times 6} = \underline{16}$$

Independent of Path

$F(x,y) = f(x,y)\hat{i} + g(x,y)\hat{j}$ is a conservative vector field in some open region D containing the pts (x_0, y_0) and (x_1, y_1) and that f & g are continuous in this region. If $F(x,y) = \nabla\phi(x,y)$ and if C starts from (x_0, y_0) and ends at (x_1, y_1) that lies in D then

$$\int_C F(x,y) \cdot dr \quad \text{or} \quad \int_C \nabla\phi \cdot dr = \phi(x_1, y_1) - \phi(x_0, y_0)$$

E doesnot depends on path

Pbms

1) Confirm that $F(x,y) = yi + xj$ is conservative then find $\int_{(0,0)}^{(1,1)} F \cdot dr$?

$\Longrightarrow$ $F(x,y) = yi + xj$

Here $f = y$ $\quad g = x$

$\frac{\partial f}{\partial y} = 1$ $\quad \frac{\partial g}{\partial x} = 1$

$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \Longrightarrow F$ is conservative

$\therefore F = \nabla \phi$ $\quad yi + xj = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y}$

$\frac{\partial \phi}{\partial x} = y \Rightarrow \phi = xy + c_1$

$\frac{\partial \phi}{\partial y} = x \Rightarrow \phi = xy + c_2$

$\therefore \underline{\underline{\phi = xy}}$

$\int_{(0,0)}^{(1,1)} F \cdot dr = \phi(1,1) - \phi(0,0) = 1 - 0 = \underline{\underline{1}}$

2) S.T $F = (2xy + z^3)i + x^2 j + 3xz^2 k$ is a conservative vector field. Find its scalar potential. Find workdone in moving an object in this field from $(1,-2,1)$ to $(3,1,4)$

$\rightarrow$ $f = 2xy + z^3$ $\quad g = x^2$ $\quad h = 3xz^2$

$$\text{curl} F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} = i\left[\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right] - j\left[\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right] + k\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)$$

$$= i(0 - 0) - j[3z^2 - 3z^2] + k[2x - 2x]$$

F is conservative vector field $= \underline{0}$

$F = \nabla\phi \Rightarrow (2xy + z^3)i + x^2 j + 3xz^2 k = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}$

$\frac{\partial \phi}{\partial x} = 2xy + z^3 \xRightarrow{\text{integ}} \phi = x^2 y + xz^3 + c_1$

$\frac{\partial \phi}{\partial y} = x^2 \Longrightarrow \phi = x^2 y + c_2$

$\frac{\partial \phi}{\partial z} = 3xz^2 \Longrightarrow \phi = xz^3 + c$

$\left.\right\}$ $\phi = x^2 y + xz^3 + c$ Potential fun

Workdone $= \int_{(1,-2,1)}^{(3,1,4)} F \cdot dr = \phi(3,1,4) - \phi(1,-2,1) = 201 - (-1)$
$= \underline{\underline{202}}$

Module II

Vector Integral theorems

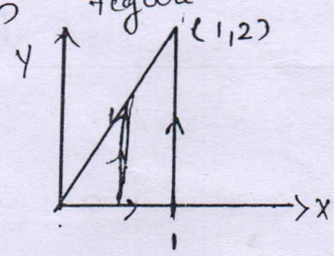
Green's Theorem

Let $\mathbb{R}$ be a simply connected plane region whose boundary is a simple, closed, piecewise smooth curve 'C' oriented counter clockwise. If $f(x,y)$ and $g(x,y)$ are continuous and have continuous first partial derivatives on some open set containing R then,

$$\int_C f(x,y)\,dx + g(x,y)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

Pbms

1) Use Green's thm evaluate $\oint x^2y\,dx + x\,dy$ along a triangular path shown in figure.



$\rightarrow$ Here $f = x^2y \qquad g = x$.

$$\frac{\partial f}{\partial y} = x^2 \qquad \frac{\partial g}{\partial x} = 1$$

$$\therefore \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 1 - x^2.$$

(0,0) to (1,2)

$$\frac{y-0}{2-0} = \frac{x-0}{1-0}$$

$y = 2x$

$$\underline{y = 2x}.$$

By Green's thm $\oint_C x^2y\,dx + x\,dy =$

$$\int_0^1\int_0^{2x} (1-x^2)\,dy\,dx$$

$$= \int_0^1 \left[y - x^2y\right]_0^{2x} dx = \int_0^1 2x - 2x^3\,dx$$

$$= \left[\frac{2x^2}{2} - \frac{2x^4}{4}\right]_0^1$$

$$= \underline{\underline{1/2}}.$$

2) Find the workdone by the force field

$$F(x,y) = [e^x - y^3]\,i + [\cos y + x^3]\,j$$

on a particle that travels once around the unit circle $x^2+y^2=1$ in the counterclockwise direction.

Work done $W = \oint_C \bar{F}\cdot d\bar{r} = \oint (e^x - y^3)\,dx + (\cos y + x^3)\,dy$

By Green's thm $= \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$.

Here $f = e^x - y^3$ $\quad g = \cos y + x^3$

$\frac{\partial f}{\partial y} = -3y^2$ $\quad \frac{\partial g}{\partial x} = 3x^2$

$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 3(x^2+y^2)$.

$= \iint_R 3(x^2+y^2)\,dA$.

Put $x^2+y^2 = r^2$ $\quad x = r\cos\theta$, $y = r\sin\theta$, $dA = r\,dr\,d\theta$ $\quad r \to 0$ to 1, $\theta \to 0$ to 2π

$$= \int_0^{2\pi}\int_0^1 3r^2 \cdot r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[3\frac{r^4}{4}\right]_0^1 d\theta = \frac{3}{4}\int_0^{2\pi} d\theta = \frac{3}{4}2\pi = \frac{3}{2}\pi.$$

3 Using Green's thm evaluate $\int_C x^2(1+y)\,dx + (x^3+y^3)\,dy$ Where C is a square bdd by $x = \pm 1$, $y = \pm 1$

$f = x^2(1+y)$ $\quad g = x^3+y^3$

$\frac{\partial f}{\partial y} = x^2$ $\quad \frac{\partial g}{\partial x} = 3x^2$.

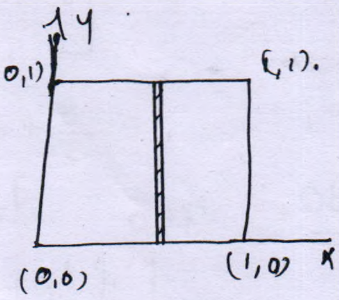
$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 2x^2$

$\therefore$ By Green's thm $\int f\,dx + g\,dy = \iint_R \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right] dA$

$$= \int_{-1}^{1}\int_{-1}^{1} 2x^2\,dx\,dy$$

$$= \int_{-1}^{1} \frac{4}{3}\,dy = 8/3$$

4. Evaluate $\oint_C y^2dx + x^2dy$ where C is a square with vertices $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$ oriented counter clockwise.



→ Here $f = y^2$ $g = x^2$

$$\frac{\partial f}{\partial y} = 2y \qquad \frac{\partial g}{\partial x} = 2x.$$

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 2(x-y)$$

Here $y \to 0$ to 1

$x \to 0$ to 1

By Green's thm $\int f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$

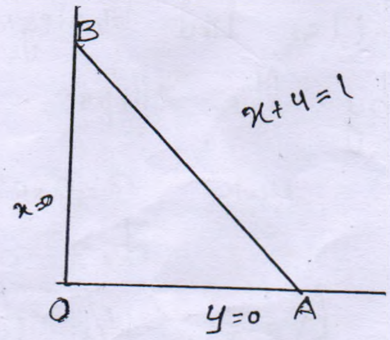
$$= \int_0^1\int_0^1 2(x-y)\,dA$$

$$2\int_0^1 \left[\frac{x^2}{2} - xy\right]_0^1 dy$$

$$= 2\int_0^1 \frac{1}{2} - y\,dy = 2\left[\frac{y}{2} - \frac{y^2}{2}\right]_0^1 = 0$$

5. Verify Green's thm in the plane $\int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$ where C is the boundary of the region defined by $x=0$, $y=0$, $x+y=1$

→ Green's thm $\int f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$



L.H.S $\int_C = \int_{OA} + \int_{AB} + \int_{BO}$.

OA. $y=0$ $dy=0$ $x \to 0$ to 1

$$\int_{OA} f\,dx + g\,dy = \int_0^1 3x^2\,dx. = \left[\frac{3x^3}{3}\right]_0^1 = 1.$$

AB $y = 1-x$ $dy = -dx$ $x \to 1$ to 0

$$\int_{AB} f\,dx + g\,dy = \int_1^0 [3x^2 - 8(1-x)^2]\,dx + [4(1-x) - 6x(1-x)]\,(-dx)$$

$$= \int_1^0 3x^2 - 8(1-x)^2 - 4(1-x) + 6x - 6x^2 \, dx$$

$$= \left[x^3 - \frac{8(1-x)^3}{-3} - \frac{4(1-x)^2}{-2} + \frac{6x^2}{2} - \frac{6x^3}{3}\right]_1^0$$

$$= \frac{8}{3} + 2 - 1 - 3 + 2 = 8/3$$

BO $x = 0 \quad dx = 0 \quad y \to 1 \text{ to } 0$

$$\int_{BO} f\,dx + g\,dy = \int_1^0 4y\,dy = \underline{\underline{-2}}$$

L.H.S $\quad \int_C f\,dx + g\,dy = 1 + \frac{8}{3} - 2 = \underline{\underline{\frac{5}{3}}}$.

R.H.S $\quad : \quad \iint \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \, dA = \int_0^1 \int_0^{1-x} (-6y + 16y) \, dy \, dx$

$$= \int_0^1 \int_0^{1-x} 10y \, dy \, dx$$

$$= \int_0^1 10 \left[\frac{y^2}{2}\right]_0^{1-x} dx$$

$$= 5 \int_0^1 [1-x]^2 \, dx = \left[5 \frac{(1-x)^3}{-3}\right]_0^1 = \underline{\underline{\frac{5}{3}}}$$

L.H.S = R.H.S

Area Using Green's theorem

$$\text{Area} = \frac{1}{2} \oint x\,dy - y\,dx.$$

Pbm

17) Use line integral to find the area enclosed by the ellipse.

$\rightarrow$ Put $x = a\cos\theta \quad y = b\sin\theta$.

$dx = -a\sin\theta\,d\theta \quad dy = b\cos\theta\,d\theta \quad \theta \to 0 \text{ to } 2\pi$

$$\text{Area} = \frac{1}{2} \oint x\,dy - y\,dx = \frac{1}{2} \int_0^{2\pi} a\cos\theta \times b\cos\theta\,d\theta - b\sin\theta \times a\sin\theta\,d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} ab\left[\cos^2\theta + \sin^2\theta\right] d\theta.$$

$$\frac{ab}{2} [\theta]_0^{2\pi} = \underline{\underline{\pi ab}}.$$

Applications of Green's theorem

1) We will use Green's thm to calculate the area bounded by the curve.
2) It is used to find the workdone of a force field.
3) Green's thm gives a relationship b/w the line integral of two dimensional vector field over a cld path in the plane and the double integral over the region it closes.

Surface Integrals

Let σ be a smooth parametric surface whose vector equation is $r = x(u,v)i + y(u,v)j + z(u,v)k$. Where (u,v) varies over a region R in the u-v plane. If $f(x,y,z)$ is continuous on σ, then

$$\iint_\sigma f(x,y,z)\,ds = \iint_R f(x(u,v), y(u,v), z(u,v)) \left\|\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right\| dA$$

1) Let σ be the surface with equation $z = g(x,y)$ and let R be its projection on the xy plane. If g has continuous first partial derivatives on R and $f(x,y,z)$ is continuous on σ then

$$\iint_\sigma f(x,y,z)\,ds = \iint_R f(x,y,g(x,y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\; dA$$

2) Surface integral over $y = g(x,z)$, $R \to$ its projection on xz plane

$$\iint_\sigma f(x,y,z)\,ds = \iint_R f(x, g(x,z), z) \sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + 1}\; dA$$

3) Surface integral over $x = g(y,z)$, R its projection of yz plane

$$\iint_\sigma f(x,y,z)\,ds = \iint_R f(g(y,z), y, z) \sqrt{\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 + 1}\; dA$$

Pbms

1) Evaluate the Surface integral $\iint_\sigma xz\, ds$ where σ is the part of the plane $x+y+z=1$ that lies in the first octant.

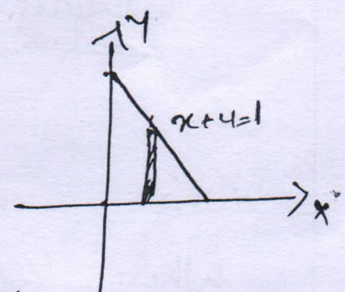
$\rightarrow$ $x+y+z=1 \Rightarrow z=1-x-y$ ie $z=f(x,y)$.

$f(x,y,z)=xz$

$$\frac{\partial z}{\partial x}=-1 \qquad \frac{\partial z}{\partial y}=-1$$

$$\therefore \iint_\sigma xz\, ds = \iint_R x(1-x-y)\cdot\sqrt{(-1)^2+(-1)^2+1}\; dA$$

$$= \iint_R (x-x^2-xy)\sqrt{3}\; dA$$



R is the projection to xy plane

$$= \sqrt{3}\int_0^1\int_0^{1-x} [x-x^2-xy]\, dy\, dx.$$

$x \rightarrow 0$ to 1

$y \rightarrow 0$ to $1-x$.

$$= \sqrt{3}\int_0^1 \left. xy - x^2y - \frac{xy^2}{2}\right]_0^{1-x} dx$$

$$\left\{ \begin{array}{l} x(1-x)-x^2(1-x) - \dfrac{x(1-x)^2}{2}. \\ = x - x^2 - x^2 + x^3 - \dfrac{x}{2} + \dfrac{2x^2}{2} - \dfrac{x^3}{2} \\ = \dfrac{x}{2} - x^2 + \dfrac{x^3}{2}. \end{array}\right.$$

$$= \sqrt{3}\int_0^1 \frac{x}{2} - x^2 + \frac{x^3}{2}\, dx$$

$$= \sqrt{3}\left[\frac{x^2}{4} - \frac{x^3}{3} + \frac{x^4}{8}\right]_0^1 = \frac{\sqrt{3}}{24}$$

---

2) Evaluate the surface integral $\iint_\sigma y^2 z^2\, ds$ where σ is the part of the cone $z=\sqrt{x^2+y^2}$ that lies b/w the planes $z=1$ & $z=2$.

$\rightarrow$ $z=f(x,y)$ $\qquad ds=\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\; dA$

$$\frac{\partial z}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}}\, 2x = \frac{x}{\sqrt{x^2+y^2}} \qquad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$ds = \sqrt{\frac{x^2+y^2}{x^2+y^2}+1}\; dA = \sqrt{2}\, dA$$

$$\iint_{\sigma} y^2 z^2 \, ds = \iint_R y^2 \sqrt{x^2+y^2} \, \sqrt{2} \, dA = \sqrt{2} \iint y^2 (x^2+y^2) \, dA$$

Where. R is the region enclosed b/w

$x^2+y^2=1$ & $x^2+y^2=4$

Using polar co-ordinates $x = r\cos\theta$, $y = r\sin\theta$.

$r \to 1$ to 2

$\theta \to 0$ to 2π

$$= \sqrt{2} \int_0^{2\pi} \int_1^2 (r\sin\theta)^2 . r^2 . r \, dr \, d\theta$$

$$= \sqrt{2} \int_0^{2\pi} \int_1^2 r^5 \sin^2\theta \, dr \, d\theta = \sqrt{2} \int_0^{2\pi} \left[\frac{r^6}{6}\right]_1^2 \sin^2\theta \, d\theta$$

$$= \frac{21}{\sqrt{2}} \int_0^{2\pi} \sin^2\theta$$

$$\frac{21}{\sqrt{2}} \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{2\pi} = \frac{21\pi}{\sqrt{2}}$$

Divergence theorem | Gauss theorem

Let G be a solid whose surface σ is oriented outward. If $F(x,y,z) = f(x,y,z)\,i + g(x,y,z)\,j + h(x,y,z)\,k$. where f, g, h have continuous first partial derivatives on some open set containing G, and if n is the outward unit normal on σ, then

$$\iint_{\sigma} F.n \, ds = \iiint_G \operatorname{div} F \, dv.$$

Flux The flux of a vector field across a closed surface with outward orientation is sometimes called the outward flux across the surface.

The outward flux of a vector field across a closed surface is equal to the triple integral of the divergence over the enclosed by the surface.

Pbms

1 Use the divergence thm to find the out-ward flux of the vector field $F(x,y,z) = z\bar{k}$ across the sphere $x^2+y^2+z^2=a^2$.

→ Let σ denote the outward-oriented spherical surface and G the region that it encloses. then $\underline{\underline{\operatorname{div} F = 1}}$

$$\boxed{\text{Flux } \phi = \iint F\cdot n \, dS}$$

By divergence thm $\boxed{\iint F\cdot n \, ds = \iiint \operatorname{div} F \, dV}$

$$\therefore \phi = \iiint dV = \text{Volume of } G = \underline{\underline{\frac{4\pi a^3}{3}}}$$

2. Use the divergence thm to find the outward flux of the vector-field $F(x,y,z) = 2xi + 3yj + z^2 k$ across the unit cube.

⟹ $\operatorname{div} F = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(3y) + \frac{\partial}{\partial z}(z^2) = \underline{\underline{5+2z}}$.

$$\text{Flux } \phi = \iint F\cdot n \, ds = \iiint \operatorname{div} F \, dv = \iiint (5+2z) dv$$

$$= \int_0^1 \int_0^1 \int_0^1 [5+2z] \, dz\, dy\, dx$$

$$= \int_0^1 \int_0^1 \left[5z + z^2\right]_0^1 dy\, dx.$$

$$= \int_0^1 \int_0^1 6 \, dy\, dx = \int_0^1 6(y)_0^1 dx.$$

$$= 6 \int_0^1 dx$$

$$= 6 (x)_0^1 = \underline{\underline{6}}$$

3 Using divergence thm. to find the outward flux of the vector field $F(x,y,z) = x^3 i + y^3 j + z^2 k$ across the surface of the region that is enclosed by the circular cylinder $x^2+y^2=9$ and the planes $z=0$ and $z=2$.

→ $\operatorname{div} F = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^2) = 3x^2 + 3y^2 + 2z$

$$\phi = \iint F\cdot n\,ds = \iiint 3x^2+3y^2+2z\ dv$$

$x = r\cos\theta$
$y = r\sin\theta$.
$dx\,dy = r\,dr\,d\theta$
$r \to 0$ to 3.
$\theta \to 0$ to 2π

$$= \int_0^{2\pi}\int_0^3\int_0^2 \left[3r^2+2z\right] r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3\int_0^2 (3r^3+2zr)\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 \left[3r^3 z + z^2 r\right]_0^2 dr\,d\theta.$$

$$= \int_0^{2\pi}\int_0^3 6r^3 + 4r\ \ dr\,d\theta.$$

$$= \int_0^{2\pi}\left(\frac{6r^4}{4} + \frac{4r^2}{2}\right)_0^3 d\theta = \int_0^{2\pi}\frac{279}{2}\,d\theta$$

$$= 279\pi$$

4 Verify divergence theorem by evaluating the surface integral and the triple integral for $F(x,y,z) = xi + yj + zk$, σ is the spherical surface $x^2+y^2+z^2=1$

→ Divergence thm $\iint F\cdot n\,ds = \iiint \operatorname{div} F\,dv$

For any point $r = xi + yj + zk$ on σ let $n = xi + yj + zk$

then $F\cdot n = (xi+yj+zk)\cdot(xi+yj+zk) = x^2+y^2+z^2 = 1$

$\therefore \iint F\cdot n\,ds = \iint ds = 4\pi$ [radius = 1]

$\operatorname{div} F = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3$

$\iiint \operatorname{div} F \, dv = \iiint 3\,dv = 3 \cdot 4/3 \pi = \underline{4\pi}$

Hence the thm

5 Use divergence thm to find flux of F across the surface σ with outward orientation for $F = (x^2+y)i + xyj - (2xz+y)k$, σ is the surface of the tetrahedron in the first octant bdd by $x+y+z=2$ and the co-ordinate planes.

$$\operatorname{div} F = \frac{\partial}{\partial x}(x^2+y) + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial z}\left(-(2xz+y)\right)$$

$$= 2x + x - 2x = \underline{x}$$

$z \to 0$ to $2-x-y$

$y \to 0$ to $2-x$

$x \to 0$ to 2.

Flux $\phi = \iint f \cdot n \, ds$

$$= \iiint \operatorname{div} F \, dv$$

$$= \int_0^2 \int_0^{2-x} \int_0^{2-x-y} x \, dz \, dy \, dx$$

$$\int_0^2 \int_0^{2-x} x(2-x-y) \, dy \, dx$$

$$= \int_0^2 \int_0^{2-x} 2x - x^2 - xy \, dy \, dx$$

$$= \int_0^2 \left[2xy - x^2 y - \frac{xy^2}{2} \right]_0^{2-x} dx$$

$$= \int_0^2 2x(2-x) - x^2(2-x) - \frac{x(2-x)^2}{2} \, dx$$

$$\frac{1}{2} \int_0^2 4x - 4x^2 + x^3 \, dx$$

$$= \frac{1}{2} \left[\frac{4x^2}{2} - \frac{4x^3}{3} + \frac{x^4}{4} \right]_0^2 = \underline{2/3}$$

SOURCES AND SINKS

A point P in an incompressible fluid is said to be a source if $(\nabla \cdot F)_p > 0$ at P and it is said to be a sink if $(\nabla \cdot F)_p < 0$

If $(\nabla \cdot F)_p = 0$ then P is free of source and sink.

Pbms

1 Determine whether the vectorfield $F = 4(x^3 - x)i + 4(y^3 - y)j + 4(z^3 - z)k$ is free of source and sink. If it is not locate them.

→ $\nabla \cdot F = 4(3x^2 - 1) + 4(3y^2 - 1) + 4(3z^2 - 1) = 12(x^2 + y^2 + z^2 - 1)$

free of source & sink $\nabla \cdot F = 0 \Rightarrow x^2 + y^2 + z^2 = 1$.

It is free of source and sink on the surface of sphere.

Source $\nabla \cdot F > 0$. ie $x^2 + y^2 + z^2 > 1$.

Sink $\nabla \cdot F < 0$ $x^2 + y^2 + z^2 < 1$.

2 Determine whether the vector field $F(x, y, z)$ is free of source and sinks. If it is not locate them.

(i). $F(x, y, z) = (y + z)i - xz^3 j + x^2 \sin y \, k$.

(ii) $F(x, y, z) = x^3 i + y^3 j + z^3 k$.

→ (i) $\nabla \cdot F = 0$ Hence no source or sinks.

(ii) $\nabla \cdot F = 3x^2 + 3y^2 + 3z^2$

$\nabla \cdot F > 0$ for all pts except at origin.

Source at all pts except at the origin.

[$\nabla \cdot F$ cannot be negative. It is has no sinks]

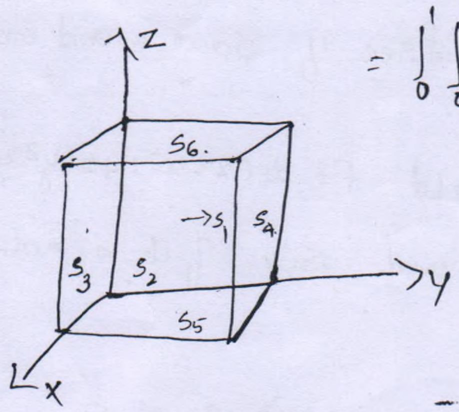
1) Use divergence thm evaluate $\iint f \cdot n \, ds$. Where $f = xi + zj + yzk$. S is the surface of the cube bdd by $x=0, x=1, y=0, y=1, z=0, z=1$. Also verify the result by computing surface integral over S.

→ $\iint f \cdot n \, ds = \iiint \operatorname{div} f \, dv$.

$\nabla \cdot f = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(yz)$

$= 1 + y$

$= \int_0^1 \int_0^1 \int_0^1 (1+y) \, dz \, dy \, dx = \frac{3}{2}$



6 Surfaces.

$S_1 \to$ yz plane $x=0$

$S_2 \to$ yz plane $x=1$

$S_3 \to$ xz plane $y=0$

$S_4 \to$ xz plane $y=1$

$S_5 \to$ xy plane $z=0$

$S_6 \to$ xy plane $z=1$

S_1 $(x=0)$

$f = zj + yzk$. $n = -i$. $f \cdot n = 0$ $\iint_{S_1} f \cdot n \, ds = 0$.

S_2 $x=1$ $f = i + zj + yzk$ $n = i$ $f \cdot n = 1$. $y \to 0$ to 1, $z \to 0$ to 1

$$\iint_{S_2} f \cdot n \, ds = \int_0^1 \int_0^1 dy \, dz = 1.$$

$S_3 = (y=0)$ $f = xi + zj$ $n = -j$ $f \cdot n = -z$

$x \to 0$ to 1. $z \to 0$ to 1

$$\iint_{S_3} f \cdot n \, ds = \int_0^1 \int_0^1 -z \, dz \, dx = -1/2$$

S_4 $(y=1)$ $f = xi + zj + yk$ $n = j$. $f \cdot n = z$. $x \to 0$ to 1. $z \to 0$ to 1

$$\iint_{S_4} f \cdot n \, ds = \int_0^1 \int_0^1 z \, dz \, dx = 1/2$$

S_5 $(z=0)$ $f = xi$ $n = -k$. $f \cdot n = 0$.

$$\iint_{S_5} f \cdot n \, ds = 0$$

$S_6 =$ $(z=1)$ $f = xi + j + yk$ $n = k$ $f \cdot n = y$ $x \to 0$ to 1, $y \to 0$ to 1

$$\iint_{S_6} f \cdot n \, ds = \int_0^1 \int_0^1 y \, dy \, dx = 1/2$$

$$\iint_S f \cdot n \, ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6} = 0 + 1 - 1/2 + 1/2 + 0 + 1/2 = \frac{3}{2}$$

Stokes theorem

Let σ be a piecewise smooth oriented surface that is bounded by a simple closed piecewise smooth curve C with positive orientation. If the components of the vector field

$$F(x,y,z) = f(x,y,z)\,i + g(x,y,z)\,j + h(x,y,z)\,k$$

are continuous and have continuous first partial derivatives on some open set containing σ, and if T is unit tangent vector to C, then

$$\oint_C F\cdot T\, ds = \iint_\sigma (\operatorname{curl} F)\cdot n\, dS.$$

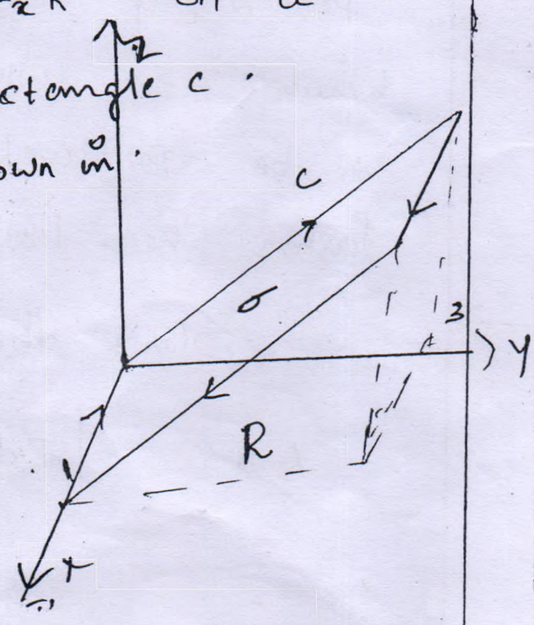
$$\text{Work} = \oint_C F\cdot dr = \iint_\sigma \operatorname{curl} F\cdot n\, dS.$$

Prob

1 Find the work performed by the force field

$$F(x,y,z) = x^2 i + 4xy^3 j + y^2 x k$$

on a particle that traverses the rectangle C in the plane $z=y$ as shown in figure.



$$\text{Work} = \int F\cdot dr$$

$$= \iint_\sigma \operatorname{curl} F\cdot n\, dS$$

By Stoke's thm

$$\text{curl}f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 4xy^3 & y^2x \end{vmatrix} = 2xyi - y^2j + 4y^3k.$$

$$\iint_\sigma f\cdot n\, ds = \iint_R f\cdot\left(\frac{\partial z}{\partial x}i + \frac{\partial z}{\partial y}j - k\right) dA$$

$$W = \iint_\sigma (\text{curl} f\cdot n)\, ds = \iint_R \text{curl}f\cdot\left(\frac{\partial z}{\partial x}i + \frac{\partial z}{\partial y}j - k\right) dA$$

$$= \iint_R (2xyi - y^2j + 4y^3k)\cdot(j - k)\, dA$$

$$= \iint_R -y^2 - 4y^3\, dy\, dx$$

$$= \int_0^1\int_0^3 (-y^2 - 4y^3)\, dy\, dx$$

$$= \int_0^1 \left[-\frac{y^3}{3} - \frac{4y^4}{4}\right]_0^3 dx = \underline{\underline{-90}}$$

2. Verify Stoke's thm for the vectorfield $f(x,y,z) = 2zi + 3xj + 5yk$, taking σ to be the portion of the paraboloid $z = 4 - x^2 - y^2$ for which $z \geq 0$ with upward orientation, and c to be positively oriented circle $x^2 + y^2 = 4$ that forms the boundary of σ in the xy plane.

→ Stokes thm $\int_c f\cdot dr = \iint_\sigma \text{curl} f\cdot n\, ds.$

L.H.S $\int f\cdot dr = \int_c [2zi + 3xj + 5yk]\cdot[dx\, i + dy\, j + dz\, k]$

$$= \int_c 2z\, dx + 3x\, dy + 5y\, dz$$

$$= \int_0^{2\pi} 3\times 2\cos t \times 2\cos t\, dt$$

[$c \rightarrow$ circle

$x = 2\cos t$, $y = 2\sin t$

$z = 0$ $\quad 0 \leq t \leq 2\pi$.

$dx = -2\sin t\, dt$

$dy = 2\cos t\, dt$ $\quad dz = 0$

$$\int_0^{2\pi} 12\cos^2 t \, dt = 12\int_0^{2\pi}\left[\frac{1+\cos 2t}{2}\right] dt$$

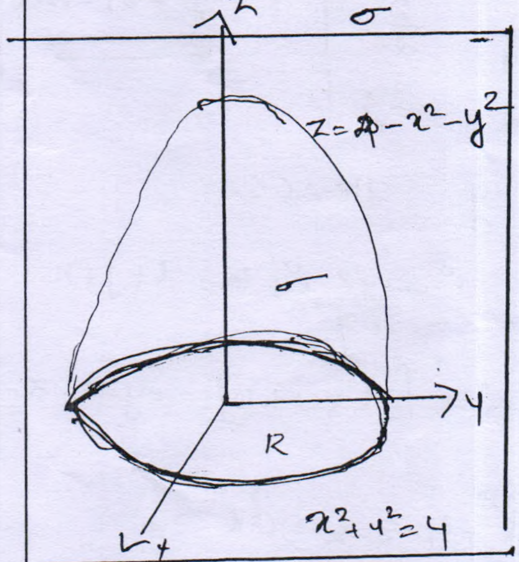
$$= 6\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi} = \underline{\underline{12\pi}}.$$

R.H.S $\quad \iint_\sigma (\text{Curl}\, F \cdot n)\, ds$

$$\text{Curl}\, F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & 3x & 5y \end{vmatrix} = 5i + 2j + 3k.$$

Here $\quad z = g(x,y) = 4 - x^2 - y^2 \quad \sigma$ Oriented up.

$$\iint_\sigma F\cdot n\, ds = \iint_R F\cdot\left(-\frac{\partial z}{\partial x} i - \frac{\partial z}{\partial y} j + k\right) dA$$



$$= \iint_R (5i + 2j + 3k)\cdot(2x i + 2y j + k)\, dA$$

$$= \iint_R (10x + 4y + 3)\, dA$$

$$= \int_0^{2\pi}\int_0^2 [10r\cos\theta + 4r\sin\theta + 3]\, r\, dr\, d\theta$$

$x = r\cos\theta$
$y = r\sin\theta$
$r \to 0$ to 2
$\theta = 0$ to 2π

$$= \int_0^{2\pi} \left[\frac{10r^3\cos\theta}{3} + \frac{4r^3}{3}\sin\theta + \frac{3r^2}{2}\right]_0^2 d\theta.$$

$$= \int_0^{2\pi} \frac{80}{3}\cos\theta + \frac{32}{3}\sin\theta + 6\, d\theta.$$

$$= \left[\frac{80}{3}\sin\theta - \frac{32}{3}\cos\theta + 6\theta\right]_0^{2\pi}$$

$$= 12\pi$$

$\therefore$ L.H.S = R.H.S $\quad$ Thm Verified

3 Use Stoke's thm to evaluate $\int F \cdot dr$. Where $F(x,y,z) = xy\,i + yz\,j + zx\,k$: C is the triangle in the plane $x+y+z=1$ with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ with a counter clockwise orientation looking from the first octant towards the origin

→ Stoke's thm $\int F \cdot dr = \iint_\sigma (\text{curl}F \cdot n)\, ds$.

$$\text{curl}F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} = \underline{\underline{-y\,i - z\,j - x\,k}}$$

$x+y+z=1 \implies z = 1-x-y$ Hence.

$$n = -\frac{\partial z}{\partial x}\,i - \frac{\partial z}{\partial y}\,j + k = i+j+k$$

$$\text{curl}F \cdot n = [-y\,i - z\,j - x\,k] \cdot [i+j+k] = -y-z-x$$

$$\iint_\sigma \text{curl}F \cdot n\, ds = \iint_R (-y-z-x)\,dA$$

$$= \iint_R -y-(1-x-y)-x\,dA$$

$$= \iint_R -dA = -\iint_R dA$$

$$= -\frac{1}{2} \times 1 \times 1 = -\tfrac{1}{2} \quad [\text{Area of a triangle}]$$

4. Consider the vector field given by the formula $F(x,y,z) = (x-z)\,i + (y-x)\,j + (z-xy)\,k$.

(i). Use Stoke's thm find the circulation around the triangle with vertices $A(1,0,0)$, $B(0,2,0)$ $C(0,0,3)$ oriented counter clockwise looking from the Origin toward the first Octant.

(ii). Find the Circulation density of F at the Origin in the direction of k

(iii) Find the unit vector n such that the Circulation density of F at the Origin is maximum in the direction of n.

(i). Equation of a plane passing through A(1,0,0) B(0,2,0) C(0,0,1) is

$2x+y+2z=2.$

$$\text{Curl}\,F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-z & y-x & z-xy \end{vmatrix} = -xi+(y-1)j-k$$

Circulation $\int F\cdot T\,ds = \iint \text{curl}F\cdot n\,ds.$

$2x+y+2z=2 \implies z = 1-x-y/2.$

$n = \frac{\partial z}{\partial x}i + \frac{\partial z}{\partial y}j - k.$

$= -i - \frac{j}{2} - k$

$\text{Curl}F\cdot n = x - \frac{(y-1)}{2} + 1 = x - y/2 + 3/2$

$$\iint_\sigma \text{curl}F\cdot n\,ds = \iint_R x - y/2 + 3/2\,dA$$

$$= \int_0^1\int_0^{2-2x} x - \frac{y}{2} + \frac{3}{2}\,dy\,dx$$

$$= \int_0^1 x+2-3x^2\,dx = \frac{3}{2}$$

(ii). Curl f at $(0,0,0) = -j - k$ also $n = k$

$\text{curl } f \cdot n = \underline{\underline{-1}}$

(iii) The rotation of f has its maximum value at the origin about the unit vector in the direction of Curl f (0,0,0)

So $n = \dfrac{-j-k}{\sqrt{1+1}} = \underline{\underline{-\dfrac{j}{\sqrt2} - \dfrac{k}{\sqrt2}}}$

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5 Use Stoke's thm to evaluate $\int_c f \cdot dr$ where $f = (x-y)i + (y-z)j + (z-x)k$ where C is the boundary of the portion of the plane $x+y+z=1$ in the first octant

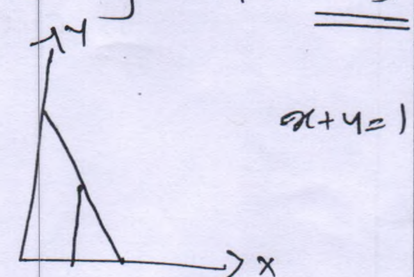
$$\int f \cdot dr = \iint \text{curl } f \cdot n \, ds.$$

$$\text{curl } f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-y & y-z & z-x \end{vmatrix} = \underline{\underline{i+j+k}}$$

$Z = 1-x-y$

$$n = -\frac{\partial z}{\partial x} i - \frac{\partial z}{\partial y} j + k = \underline{\underline{i+j+k}}$$

$$\text{curl } f \cdot n = [i+j+k] \cdot [i+j+k] = 1+1+1 = \underline{\underline{3}}$$



$x \to 0$ to 1

$y \to 0$ to $\underline{\underline{1-x}}$

$$\iint \text{curl } f \cdot n \, ds = \int_0^1 \int_0^{1-x} 3 \, dA = 3 \int_0^1 (1-x)\, dx$$

$$= 3 \left[x - \frac{x^2}{2}\right]_0^1 = 3[1 - 1/2]$$

$$\underline{\underline{= 3/2}}$$

Theorems

(1) Green's thm $\rightarrow \int_C f(x,y)\,dx + g(x,y)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$

Relation b/w line integral & Double integral.

Applications $\rightarrow$ (1) find area of a plane region.

(2) Gauss Divergence theorem $\rightarrow \iint_\sigma F\cdot n\,ds = \iiint \operatorname{div} F\,dv$

Relation b/w Surface integral and triple integral.

Applications $\rightarrow$ (1). used in electrostatic fields $\rightarrow$ Gauss' law

$\text{Flux} = \iint F\cdot n\,ds = \iiint \operatorname{div} F\,dv$

(2) find Volume.

(3) Source & Sink.

(3) Stoke's thm $\rightarrow \int F\cdot dr = \iint_\sigma \operatorname{curl} F\cdot n\,ds$

Relation b/w line integral and double integrals

Applications $\rightarrow$ (1) find work.

(2) This thm is used in physics, especially in electromagnetism.

MAT 102
Module III

1

...neous Linear ODE's

A second order ODE is called linear if it can be written as $\boxed{y'' + P(x)y' + q(x)y = r(x)}$ → (2)

and nonlinear if it cannot be written in the above form. This equation is linear in y and its derivatives, where as P, q, r be any function of x.

If $r(x) = 0$ in, $y'' + P(x)y' + q(x)y = 0$ it is called homogeneous.

If $r(x) \neq 0$, it is called nonhomogeneous.

Exa:- $y'' + 3y' + y = 0$ → homogeneous

$y'' + 25y' = e^{-x}\cos x$ → Non homogeneous

An ODE of n^{th} order is called linear if it can be written as

$$\boxed{y^{(n)} + P_{n-1}(x)\, y^{(n-1)} + \cdots + P_1(x)y' + P_0(x)\, y = r(x)} \rightarrow (3)$$

The coefficients $P_0, P_1, P_2 \ldots P_{n-1}$ be any functions of x. The above form (3) is called nonlinear.

As before, if $r(x) = 0$ it is called homogeneous and if $r(x) \neq 0$ it is called Non homogeneous.

A Solution of n^{th} order ODE in some open interval I is a function $y = h(x)$, that is defined and n times differentiable on I and such that ODE becomes an identity if we replace y by h, y' by derivative of h etc.

If y_1, y_2 are two functions of x, $c_1 y_1 + c_2 y_2$ is called a linear combination of y_1, y_2.

If $y_1, y_2, \dots y_n$ are functions of x $c_1 y_1 + c_2 y_2 + \dots +$ is called Linear combination of $y_1, y_2, y_3 \dots y_n$

Fundamental theorem for homogeneous linear ODE [Superposition Principle or Linearity principle]

(1) For a homogeneous linear ODE of 2^{nd} order, an l.c of solutions in open interval I is again a solution on I. In particular for such an equation Sums and constant multiples of solutions are again solutions.

(2) For a homogeneous linear ODE of n^{th} order any L.C of the solutions in open interval I is again a solution in I. In particular sum and constant multiples of solutions are again Solutions.

Note:- The above theorem does not hold for non homogeneous linear ODEs.

Exa:- $y'' + y = 1$ (Non homogeneous)

$1 + \cos x$, $1 + \sin x$ are solutions of $y'' + y = 1$. But their Sum is not a solution (2).

$\therefore$ $y = 1 + \cos x + 1 + \sin x$

$y = 2 + \sin x + \cos x$

$y' = \cos x - \sin x$

$y'' = -\sin x - \cos x$

$\therefore y'' + y = -\sin x - \cos x + 2 + \sin x + \cos x = 2 \neq 1$

Exa: $\cos x$, $\sin x$ are solutions of $y'' + y = 0$.

$a\cos x + b\sin x$ is again a solution of $y'' + y = 0$.

exa: e^{-x} and e^{x} are solutions of $y'' - y = 0$

Definition :- Two functions y_1, y_2 of x are Said to be linearly independent if $k_1 y_1 + k_2 y_2 = 0 \Rightarrow$ $k_1 = 0$, $k_2 = 0$

y_1, y_2 are linearly dependent if $k_1 y_1 + k_2 y_2 = 0$ for some constants k_1, k_2 not both zero.

If y_1, y_2 are L.D, y_1, y_2 are proportional. If y_1, y_2 are L.I they are not proportional.

Definition:- A general solution of a homogeneous linear ODE of 2^{nd} order on an open interval I is a solution $\boxed{y = c_1 y_1 + c_2 y_2}$ where y_1, y_2 are solutions of the ODEs on I that are not proportional. ie, y_1, y_2 are L.I and c_1, c_2 are arbitrary consts. These y_1, y_2 are called a basis of solutions of the equation on I. A particular solution of (1) on I is obtained if we assign specific values to C_1, C_2

Definition:- If $y_1, y_2 \cdots y_n$ are functions of x defined on an open interval I, they are said to be L.I. if $k_1 y_1 + k_2 y_2 + \cdots + k_n y_n = 0$ holds on I for some $k_1, k_2 \cdots$ not all zero, they are said to be L.D.

If $y_1, y_2 \cdots y_n$ are linearly dependent, we can express atleast one of these function as a L.C of other $(n-1)$ functions.

$k_1 y_1 + k_2 y_2 + \cdots + k_n y_n = 0$ suppose $k_1 \neq 0$,

$$\boxed{y_1 = -\frac{k_2}{k_1} y_2 - \cdots - \frac{k_n}{k_1} y_n}$$ ie, l.c of $y_1, y_2, \ldots y_n$.

Problems:-

S.T $y_1 = x^2$ $y_2 = 5x$ $y_3 = 2x$ are Linearly dependent on any interval

Ans: $1 \cdot y_2 + 0 \cdot y_1 - \frac{5}{2} y_3 = 0$

So we can find constants k_1, k_2, k_3 such that $k_1 \neq 0$ $k_3 \neq 0$ so y_1, y_2, y_3 are L.D

S.T $y_1 = x$, $y_2 = x^2$, $y_3 = x^3$ are L.I on any interval say $-1 \leq x \leq 2$

Ans:- Let $k_1 x + k_2 x^2 + k_3 x^3 = 0$ ——— (1)

Let $x=-1$, $x=1$, $x=2$

$x=-1$ in (1) gives $-k_1+k_2-k_3=0$ ——— (2)

$x=1$ in (1) gives $k_1+k_2+k_3=0$ ——— (3)

$x=2$ in (1) gives $2k_1+4k_2+8k_3=0$. ——— (4)

Solving (2), (3) & (4) we have $k_1=k_2=k_3=0$

Verify that $y_1=e^x$ $y_2=e^{-x}$ are solutions of the O.D.E $y''-y=0$ $y(0)=6$, $y'(0)=-2$ Write the general solution, basis and particular solution.

$y_1=e^x$

$y_1'=e^x$

$y_1''=e^x$

$y''-y=e^x-e^x=0$

$\therefore y_1=e^x$ is a solution

$y_2=e^{-x}$

$y_2'=-e^{-x}$

$y_2''=e^{-x}$

$y''-y=e^{-x}-e^{-x}=0$

$\therefore y_2=e^{-x}$ is a solution.

Let $k_1e^x+k_2e^{-x}=0$

$k_2e^{-x}=-k_1e^x$

$e^{-2x}=\frac{-k_1}{k_2}=\text{const}$

But e^{-2x} is not const. $\therefore y_1=e^x$ & $y_2=e^{-x}$ is not proportional. Hence $y_1=e^x$ & $y_2=e^{-x}$ are linearly independent

So $y = c_1 e^{x} + c_2 e^{-x}$ is a general solution.

$y(0) = 6 \Rightarrow 6 = c_1 + c_2$ ——— (1)

$y'(x) = c_1 e^{x} - c_2 e^{-x}$

$y'(0) = -2 \Rightarrow -2 = c_1 - c_2$ ——— (2)

Solving (1) & (2) we get $c_1 = 2$ $c_2 = 4$

ie, $y = 2e^{x} + 4e^{-x}$ is a particular solution

e^{x}, e^{-x} form a basis

Definition:-

1. An initial value problem for the 2^{nd} order homogen linear ODE $y'' + P(x) y' + q(x) y = r(x)$ ——— (1) consists of (1) and the initial conditions $y(x_0) = k$, $y'(x_0) = k_1$ with given k_0, k_1 and x_0 is in the open interval I considered.

An initial value problem for n^{th} order homogeneo linear ODE consists of the ODE and the n initial conditions $y(x_0) = k_0$, $y'(x_0) = k_1, \dots y^{n-1}(x_0) = k_{n-1}$ with x_0 in the open interval I considered an given $k_0, k_1, \dots k_{n-1}$

Existence & uniqueness of solution of homogeneous linear ODEs.

Theorem:- If $P(x)$ and $q(x)$ are continuous fun on some open interval I and x_0 is in I

then the initial value problem $y''+p(x)y'+q(x)y=0$ with initial conditions $y(x_0)=k_0$, $y'(x_0)=k_1$ has a unique solution $y(x)$ on the interval I

Theorem:-

If the coefficients $P_1(x), P_2(x) \dots P_{n-1}(x)$ are continuous on some open interval I and x_0 is in I then the initial value problem, $y^{n}+P_{n-1}(x)\, y^{n-1}+ \dots + P_1(x)y'+P_0(x)y=0$ with initial conditions $y(x_0)=k_0$ $y'(x_0)=k_1 \dots y^{n-1}(x_0)=k_{n-1}$ has a unique solution $y(x)$ on I

Definition:- Wronskian.

The Wronskian W of two solutions y_1, y_2 is defined as
$$W(y_1,y_2)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

The Wronskian W of n solutions $y_1, y_2, y_3 \dots y_n$ is defined as
$$W(y_1,y_2\dots y_n)=\begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \vdots & & & \\ y_1^{n-1} & y_2^{n-1} & \dots & y_n^{n-1} \end{vmatrix}$$

f $W \neq 0$ then y_1, y_2 are L.I ② If $W=0$ then y_1, y_2 are L.D

heorem:-

Let $y''+P(x)y'+q(x)y=0$ ——(1) has continuous efficients $p(x), q(x)$ on an open interval I then the two solutions y_1, y_2 of (1) are linearly dependent on I iff the wronskian $W(y_1, y_2)$ is zero at some x_0 on I. Further if $W=0$, at an x_0 in I then $W=0$ on I. Hence if there is an x_0 in I at which $W \neq 0$, then y_1, y_2 are L.I on I.

Exa: I The functions $y_1 = \cos wx$, $y_2 = \sin wx$ are Solutions of $y'' + w^2 y = 0$

The wronskian $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$$= \begin{vmatrix} \cos wx & \sin wx \\ -w\sin wx & w\cos wx \end{vmatrix}$$

$$= w(\cos^2 wx + \sin^2 wx)$$
$$= w$$

So these solutions are linearly independent iff w

Exa: 2 S.T $e^{-2x}, e^{-x}, e^{x}, e^{2x}$ are linearly independ

$$W = \begin{vmatrix} e^{-2x} & e^{-x} & e^{x} & e^{2x} \\ -2e^{-2x} & -e^{-x} & e^{x} & 2e^{2x} \\ 4e^{-2x} & e^{-x} & e^{x} & 4e^{2x} \\ -8e^{-2x} & -e^{-x} & e^{x} & 8e^{2x} \end{vmatrix}$$

$$= e^{-2x} \cdot e^{-x} \cdot e^{x} \cdot e^{2x} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 1 & 2 \\ 4 & 1 & 1 & 4 \\ -8 & -1 & 1 & 8 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 3 & 4 \\ 4 & -3 & -3 & 0 \\ -8 & 7 & 9 & 16 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 3 & 4 \\ -3 & -3 & 0 \\ 7 & 9 & 16 \end{vmatrix}$$

$$= 3\begin{vmatrix} 3 & 4 \\ 9 & 16 \end{vmatrix} - 3\begin{vmatrix} 1 & 4 \\ 7 & 16 \end{vmatrix}$$

$$= 3\,(48-36) - 3(16-28)$$

$$= 3\cdot 12 + 3\cdot 12$$

$$= 72 \neq 0$$

So they are <u>linearly</u> independent.

<u>Theorem:</u> [<u>Existence of General solution</u>]

1) If $p(x)$, $q(x)$ are continuous on an open interval I, then $y'' + p(x)y' + q(x)y = 0$ has a general solution on I

2) If the coefficients $P_0(x)$, $P_1(x) \dots P_{n-1}(x)$ of the ODE $y^n + P_{n-1}(x)\, y^{n-1} + \dots + P_0(x)y = 0$ ——— (2) are continuous on some open interval I then (2) has a general solution on I

Theorem:- (General solution includes all solutions)

1) If the ODE $y'' + p(x)y' + q(x)y = 0$ ——— (1) have continuous coefficients $p(x)$, $q(x)$ on some open

interval I, then every solution $y = Y(x)$ on I is of the form $Y(x) = c_1 y_1(x) + c_2 y_2(x)$ where y_1, y_2 are basis of solutions of (1) on I and c_1, c_2 are suitable constants. Hence the equation does not have singular solution. ie, solutions not obtainable from general solution.

Problems:

I verify that the given functions are L.I and form a basis of solutions of the given ODE,

$y'' + 9y = 0$, $y(0) = 2$ $y'(0) = -1$

$\cos 3x$, $\sin 3x$

$y = \cos 3x$

$y' = -3\sin 3x$

$y'' = -9\cos 3x$

$y'' + 9y = -9\cos 3x + 9\cos 3x$

$= 0$

$\therefore \cos 3x$ is a solution

$y = \sin 3x$

$y' = 3\cos 3x$

$y'' = -9\sin 3x$

$y'' + 9y = -9\sin 3x + 9\sin 3x$

$= 0$

$\therefore \sin 3x$ is a solution

$$W = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix}$$

$= 3\cos^2 3x + 3\sin^2 3x = 3 \neq 0$

$\therefore \cos 3x$ & $\sin 3x$ are L.I

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So $y = c_1 \cos 3x + c_2 \sin 3x$ is a general solution.

$y(0) = 2 \Rightarrow 2 = c_1$

$y'(0) = -1 \Rightarrow -1 = 3c_2$

$c_2 = -1/3$

$\therefore y = 2\cos 3x - \frac{1}{3}\sin 3x$ is a particular solution. So $\cos 3x, \sin 3x$ form a basis.

$y'' + 2y' + 2y = 0$, $y(0) = 0$, $y'(0) = 15$ $e^{-x}\cos x$, $e^{-x}\sin x$

$y = e^{-x}\cos x$

$y' = -e^{-x}\sin x + -\cos x\, e^{-x}$

$y'' = -[e^{-x}\cos x + -\sin x\, e^{-x} + -e^{-x}\sin x + -e^{-x}\cos x]$

$= 2e^{-x}\sin x$

$y'' + 2y' + 2y = 2e^{-x}\sin x - 2e^{-x}\sin x - 2e^{-x}\cos x + 2e^{-x}\cos x$

$= 0$

$\therefore e^{-x}\cos x$ is a solution

$y = e^{-x}\sin x$

similarly $y = e^{-x}\sin x$ is also a solution.

$e^{-x}\cos x$ and $e^{-x}\sin x$ are L.I because,

$k_1 e^{-x}\cos x + k_2 e^{-x}\sin x = 0$

$\tan x = -\frac{k_1}{k_2} = a$ const . But $\tan x$ is not constant.

So $e^{-x}\cos x$ & $e^{-x}\sin x$ are not proportional

$\therefore$ L.I

So $y = c_1 e^{-x}\cos x + c_2 e^{-x}\sin x$ is a general sol

$y(0) = 0 \implies 0 = c_1$

$y'(x) = c_1\left[-e^{-x}\sin x + -e^{-x}\cos x\right] + c_2\left[e^{-x}\cos x - e^{-x}\sin\right.$

$y'(0) = 15 \implies c_1(0-1) + c_2(1-0)$

$\therefore 15 = c_2$

$\therefore y = 15e^{-x}\sin x$ is a particular solution

$\therefore e^{-x}\sin x$ & $e^{-x}\cos x$ form a basis

Home work

1. $y'' + 2y' + y = 0$ $y(0) = 2$, $y'(0) = 1$, e^{-x}, xe^{-x}

2. S.T the given functions are solutions and fo a basis on any interval. Use wronskians.

(i) $e^{x}, e^{-x}, e^{x/2}$, $2y''' - y'' - 2y' + y = 0$.

(ii) $\cos x, \sin x, x\cos x, x\sin x$ $y^{iv} + 2y'' + y = 0$

$\pm i, \pm i$ $\therefore y = (c_1 + c_2 x)(c_3\cos x + c_4\sin x)$

(iii) $e^{-4x}, xe^{-4x}, x^2e^{-4x}$, $y''' + 12y'' + 48y' + 64y = 0$

Homogeneous Linear ODEs (7) With Const. Coefficients

Consider 2^{nd} order homogeneous linear ODE

$y'' + ay' + by = 0$ where a, b are constants.

The characteristic equation or auxiliary equation is $\lambda^2 + a\lambda + b = 0$. Let the characteristic roots be λ_1 & λ_2

Case I :- If λ_1, λ_2 are real distinct the general solution is,

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

Case II :- If λ is a real double root, the general solution is,

$$y = (c_1 + c_2 x) e^{\lambda x}$$

Case III :- If the roots are $\alpha \pm i\beta$, the general solution is, $y = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x]$

Consider the n^{th} order homogeneous linear ODE,

$$y^n + a_{n-1} y^{n-1} + \cdots + a_1 y' + a_0 = 0$$

The characteristic equation is, $\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$

Case-I : If all roots of auxiliary equation are distinct, say $\lambda_1, \lambda_2 \ldots \lambda_n$ the general soln is,

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \cdots + c_n e^{\lambda_n x}$$

Case-II :- If two roots of the auxiliary eqn are equal, $\lambda_1 = \lambda_2 = \lambda$ the solution is,

$$y = (c_1 + c_2 x) e^{\lambda x} + c_3 e^{\lambda_3 x} + \cdots + c_n e^{\lambda_n x}$$

Case III

If two roots of A·E are complex number say, $\lambda_1 = \alpha + i\beta$ $\lambda_2 = \alpha - i\beta$ solution is,

$$y = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x] + c_3 e^{\lambda_3 x} + c_4 e^{\lambda_4 x} + \cdots + c_n e^{\lambda_n x}$$

Case IV :- If two pairs of imaginary roots equal, solution is,

$$y = e^{\alpha x}\left[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x\right] + c_5 e^{\lambda_5 x} + \cdots + c_n e^{\lambda_n x}.$$

Problems:

1. Solve the initial value problem $y'' + y' - 2y = 0$

$y(0) = 4$ $y'(0) = 5$

Auxiliary eqn is, $\lambda^2 + \lambda - 2 = 0$

$$\lambda = \frac{-1 \pm \sqrt{1+4\cdot 2}}{2} = 1, -2$$

$$y = c_1 e^x + c_2 e^{-x}$$

$$\therefore \quad y(x) = c_1 e^x + c_2 e^{-2x}$$

$y(0) = 4 \implies 4 = c_1 + c_2$

(8)

$$y'(x) = c_1 e^{x} - 2c_2 e^{-2x}$$

$$y'(0) = -5 \implies c_1 - 2c_2 = -5 \quad \text{——} \ (2)$$

Solving (1) and (2) we have $c_1 = 1$ $c_2 = 3$

$$\therefore \ y = e^{x} + 3e^{-2x}$$

Solve the initial value problem $y'' - 4y' + 4y = 0$

$y(0) = 3, \quad y'(0) = 1$

Ans:- The A.E is $\lambda^2 - 4\lambda + 4 = 0$

$$(\lambda - 2)^2 = 0$$

$$\lambda = 2, 2$$

$$\therefore \ y(x) = (c_1 + c_2 x)\, e^{2x}$$

$$y(0) = 3 \implies 3 = c_1 \quad \text{——} \ (1)$$

$$y'(x) = 2(c_1 + c_2 x)\, e^{2x} + e^{2x} \cdot c_2$$

$$y'(0) = 1 \implies 1 = 2c_1 + c_2 \quad \text{——} \ (2)$$

Solving (1) and (2) $\quad c_2 = -5$

$$\therefore \ y = (3 - 5x)\, e^{2x}$$

Extra problems:-

I. Solve the following initial value problems.

(1) $y'' + y' - 6y = 0 \qquad y(0) = 10, \quad y'(0) = 0$

2) $y'' + 4y' + 4y = 0 \qquad y(0) = 1 \quad y'(0) = 1$

3) $y'' - y = 0 \qquad y(0) = 3, \quad y'(0) = -3$

4) $4y'' - 3y = 0, \quad y(0) = 0 \quad y'(0) = -5$

5 $4y'' - 4y' - 3y = 0, \quad y(-2) = e, \quad y'(-2) = \frac{e}{2}$

6 $8y'' - 2y' - y = 0 \quad y(0) = -0.2 \quad y'(0) = -0.325$

Solve the following:-

1. $y'' + 9y' + 20y = 0$
2. $y'' - 2y' + 2y = 0$
3. $y'' - 25y = 0$
4. $y'' + 9y = 0$
5. $y'' - 10y' + 25y = 0$
6. $y'' + 6y' + 25y = 0$.

Solve the following:-

1. $y''' - 7y' - 6y = 0$

A.E is, $\lambda^3 - 7\lambda - 6 = 0$

$\lambda = -1$ is a root. (by trial method)

$$\lambda+1 \,\big)\, \lambda^3 - 7\lambda - 6 \,\big(\, \lambda^2 - \lambda - 6$$
$$\lambda^3 + \lambda^2$$
$$-\lambda^2 - 7\lambda$$
$$-\lambda^2 - \lambda$$
$$-6\lambda - 6$$
$$-6\lambda - 6$$
$$0$$

$\therefore (\lambda+1)(\lambda^2 - \lambda - 6) = 0$

$(\lambda+1)(\lambda-3)(\lambda+2) = 0$

$\lambda = -1, -2, 3$

$\therefore y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x}$.

Second Order Euler - Cauchy Eqn

Euler - Cauchy eqns are ODEs of the form

$x^2y'' + axy' + by = 0.$ → (1) With given constants. a and b ~~are~~. and unknown function $y(x)$

Substitute $y = x^m$

then $y' = mx^{m-1}$

$y'' = m(m-1)x^{m-2}$.

(1) ⇒ $x^2 . m(m-1)x^{m-2} + ax . mx^{m-1} + bx^m = 0.$

$x^m[m(m-1) + am + b] = 0$

A.E $m^2 + (a-1)m + b = 0.$

$m_1 = \frac{1}{2}(1-a) + \sqrt{\frac{1}{4}(1-a)^2 - b}$

$m_2 = \frac{1}{2}(1-a) - \sqrt{\frac{1}{4}(1-a)^2 - b}$

Case I Roots are real & diff.

$y = c_1x^{m_1} + c_2x^{m_2}.$

Case II Real and equal

$y = (c_1 + c_2 \ln x)x^m.$

Case III Imaginary $\alpha + i\beta$ $y = x^{\alpha}[c_1\cos(\beta\ln x) + c_2\sin(\beta\ln x)]$

Pblms

1 Solve $x^2y'' + 0.6xy' + 16.04y = 0.$

→ ~~A.E~~ $y = x^m$ $y' = mx^{m-1}$ $y'' m(m-1)x^{m-2}$

A.E $m^2 + (0.6-1)m + 16.04 = 0$

$m^2 - 0.4m + 16.04 = 0$

$m_1 = 0.2 + 4i \qquad m_2 = 0.2 - 4i$

$x = e^{\ln x}$

$$x^{m_1} = x^{0.2+4i} = x^{0.2}(e^{\ln x})^{4i} = x^{0.2} e^{(4\ln x)i}$$

$$= x^{0.2}\left[\cos(4\ln x) + i\sin(4\ln x)\right]$$

$$x^{m_2} = x^{0.2}\left[\cos(4\ln x) - i\sin(4\ln x)\right]$$

Solution $\quad y = x^{0.2}\left[A\cos(4\ln x) + B\sin(4\ln x)\right]$

2 $\quad x^2y'' - 2y = 0$

$\rightarrow \quad y = x^m \quad y' = mx^{m-1} \quad y'' = m(m-1)x^{m-2}$

$m(m-1)x^{m-2} \cdot x^2 - 2x^m = 0$

AE $\quad (m^2 - m - 2) = 0. \Rightarrow (m-2)(m+1) = 0$

$m = 2, \; m = -1$

$$y = C_1 x^2 + C_2 x^{-1}$$

3, $\quad 5x^2y'' + 23xy' + 16.2y = 0 \qquad x^2y'' + \frac{23}{5}xy' + \frac{16.2}{5}y = 0$

A.C $\quad m^2 + \left(\frac{23}{5} - 1\right)m + \frac{16.2}{5} = 0$

$m^2 + \cancel{22m}\,3.6m + 16.2 = 0 \qquad m =$

$m^2 + 3.6m + 3.24 = 0$

$m = -1.8, -1.8$

$$y = (C_1 + C_2\ln x)\,x^{-1.8}$$

Non homogeneous ODE

Consider the second order nonhomogeneous linear ODE $y'' + p(x)y' + q(x)y = r(x)$, where $r(x) \neq 0$ solution $y(x) = y_h(x) + y_p(x)$.

here $y_h = c_1 y_1 + c_2 y_2$ is a solution of corresponding homogeneous ODE and y_p is the particular soln.

Method of undetermined coefficients

$$y'' + ay' + by = r(x).$$

Solution $y(x) = y_h(x) + y_p(x)$.

1) Basic Rule : If $r(x)$ is one of the function in 1st column choose y_p corresponding to it.

Terms in $r(x)$	Choice for $y_p(x)$.
$ke^{r(x)}$	$ce^{r(x)}$
$kx^n \quad (n=0,1,2,\cdots)$	$k_n x^n + k_{n-1}x^{n-1} + \cdots + k_1 x + k_0$.
$k\cos\omega x$, $k\sin\omega x$	$K\cos\omega x + M\sin\omega x$.
$ke^{\alpha x}\cos\omega x$, $ke^{\alpha x}\sin\omega x$	$e^{\alpha x}\left[K\cos\omega x + M\sin\omega x\right]$

2) Modification Rule : If a term in your choice for y_p happens to be a solution of the homogeneous ODE corresponding to eqn. multiply this term

by x [or by x^2 if this solution corresponds to a double root of the characteristic equation of the homogeneous ODE]

3) Sum Rule : If $r(x)$ is a sum of functions then choose for y_p the sum of the functions in the corresponding lines of the second column

Pbms

1 Solve the initial value pbm $y'' + y = 0.001x^2$ ——— (1)

$y(0) = 0 \qquad y'(0) = 1.5$

→ $D^2 y + y = 0. \qquad (D^2+1)y = 0$

A.E $\quad \lambda^2 + 1 = 0 \quad \lambda^2 = -1 \quad \lambda = \pm i$.

$$\boxed{y_h(x) = A\cos x + B\sin x}$$

$y_p = k_2 x^2 + k_1 x + k_0$

$y_p' = 2k_2 x + k_1 \qquad y_p'' = 2k_2$.

(1) $\Rightarrow \quad 2k_2 + k_2 x^2 + k_1 x + k_0 = 0.001x^2$

Compare coefficients

$\underline{\underline{k_2 = 0.001}} \qquad k_1 = 0. \qquad 2k_2 + k_0 = 0$

$k_0 = -2k_2$
$= \underline{\underline{-0.002}}$

$\underline{\underline{y_p = 0.001x^2 - 0.002}}$.

$\therefore y = y_h + y_p = A\cos x + B\sin x + 0.001x^2 - 0.002$

$x=0 \quad y=0$

$\Rightarrow A - 0.002 = 0 \quad \Rightarrow \underline{\underline{A = 0.002}}$

$y' = -A\sin x + B\cos x + 0.002x$

$x=0 \quad y' = 1.5$

$\Rightarrow B = 1.5$

$\therefore \underline{\underline{y = 0.002\cos x + 1.5\sin x + 0.001x^2 - 0.002}}$

2. Solve the IVP $y'' + 3y' + 2.25y = -10e^{-1.5x}$.

$y(0)=1, \quad y'(0)=0.$

$\rightarrow$ Step 1 A.E $\lambda^2 + 3\lambda + 2.25 = 0 \Rightarrow \lambda = -1.5, -1.5$

Soln $y_h(x) = [C_1 + C_2 x]e^{-1.5x}$.

Step 2.

$r(x) = -10e^{-1.5x}$.

$e^{-1.5x}$ is part of $y_h(x)$ [-1.5 Repeated root]

$y_p = cx^2 e^{-1.5x}$.

$y_p' = c[2xe^{-1.5x} - 1.5x^2 e^{-1.5x}]$

$= ce^{-1.5x}[2x - 1.5x^2]$

$y_p'' = ce^{-1.5x}[2 - 6x + 2.25x^2]$

Sub in given eqn.

$$ce^{-1.5x}\left[2-6x+2.25x^2\right] + 3ce^{-1.5x}\left[2x-1.5x^2\right] + 2.25\,cx^2e^{-1.5x}$$
$$= -10e^{-1.5x}.$$

Cancel $e^{-1.5x}$.

$$c\left[2-6x+2.25x^2\right] + 3c\left[2x-1.5x^2\right] + 2.25\,cx^2 = -10$$

Compare Coefficent of x^2

$$2.25c - 4.5c + 2.25c = 0 \qquad 0=0$$

Coefficent of x. $\quad -6c + 6c = 0 \qquad 0=0$

Constant term $\quad 2c = -10 \qquad \underline{\underline{c = -5}}$

$$\therefore y = y_h + y_p = \underline{\underline{(c_1+c_2x)e^{-1.5x} - 5x^2e^{-1.5x}}}$$

Step 3

$x=0 \qquad y=1$

$$\underline{\underline{c_1 = 1}}.$$

$$y' = c_2e^{-1.5x} + (c_1+c_2x)\times -1.5e^{-1.5x} - \left[5x^2\times -1.5e^{-1.5x} + e^{-1.5x}\times 10x\right]$$

$x=0 \quad y'=0$

$$\Rightarrow c_2 - 1.5c_1 = 0.$$

$c_2 = 1.5c_1 \qquad \underline{\underline{c_2 = 1.5}}$

$$\therefore \underline{\underline{y = (1+1.5x)e^{-1.5x} - 5x^2e^{-1.5x}}}$$

3 Solve $y'' + 2y' + 0.75y = 2\cos x - 0.25\sin x + 0.09x$.

$y(0) = 2.78$, $y'(0) = -0.43$.

Step 1

A.E $\lambda^2 + 2\lambda + 0.75 = 0 \Rightarrow \lambda = -1/2, -3/2$.

$$y_h = c_1 e^{-1/2 x} + c_2 e^{-3/2 x}$$

Step 2

$y_p = y_{p_1} + y_{p_2}$. y_{p_1} for $2\cos x - 0.25\sin x$

y_{p_2} for $0.09x$.

$$\left.\begin{aligned} y_{p_1} &= K\cos x + M\sin x \\ y_p' &= -K\sin x + M\cos x \\ y_p'' &= -K\cos x - M\sin x \end{aligned}\right\} \rightarrow (2) \qquad \left.\begin{aligned} y_{p_2} &= k_1 x + k_2 \\ y_{p_2}' &= k_1 \\ y_{p_2}'' &= 0 \end{aligned}\right\} \rightarrow (3)$$

(2) Subs (2) in (1)

$-K\cos x - M\sin x + 2[-K\sin x + M\cos x] + .75[K\cos x + M\sin x]$

$= 2\cos x - 0.025\sin x + 0.09x$.

Compare Coefficent of $\cos x$, $\sin x$

$-K + 2M + .75K = 2$. $\quad -M + -2K + .75M = -0.25$

$\Rightarrow K = 0 \quad M = 1$

Sub (3) in (1)

$+2[k_1] + 0.75[k_1 x + k_2] = 2\cos x - 0.25\sin x$

$+ 0.09x$.

Compare Coefficent of x $\quad 0.75k_1 = 0.09. \Rightarrow k_1 = 0.12$

Constant $\quad 2k + 0.75k_2 = 0 \Rightarrow k_2 = -0.32$

$$\therefore y = c_1 e^{-x/2} + c_2 e^{-3x/2} + \sin x + 0.12x - 0.32$$

Step 3

$x = 0 \quad y = 2.75$

$$2.75 = c_1 + c_2 - 0.32 \longrightarrow (4).$$

$$y' = -\tfrac{1}{2} c_1 e^{-x/2} - \tfrac{3}{2} c_2 e^{-\frac{3x}{2}} + \cos x + 0.12$$

$x = 0 \quad y' = -0.43$

$$-0.43 = -\tfrac{1}{2} c_1 - \tfrac{3}{2} c_2 + 1 + 0.12 \longrightarrow (5)$$

(3) & (4) $\Rightarrow c_1 = 3.1$ & $c_2 = 0$

$$\therefore y = 3.1 e^{-x/2} + \sin x + 0.12x - 0.32$$

4 Solve the IVP $y''' + 3y'' + 3y' + y = 30e^{-x} \longrightarrow (1)$

$y(0) = 3 \qquad y'(0) = -3 \qquad y''(0) = -47$

$\rightarrow$ **Step 1**

A.E $\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0 \Rightarrow \lambda = -1, -1, -1$

$$y_h = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}.$$

$$= (c_1 + c_2 x + c_3 x^2) e^{-x}.$$

Step 2

$$y_p = c x^3 e^{-x}.$$

$$y_p' = 3cx^2 e^{-x} - cx^3 e^{-x} = c(3x^2 - x^3) e^{-x}.$$

$$y_p'' = 6cxe^{-x} - 3cx^2 e^{-x} - 3cx^2 e^{-x} + cx^3 e^{-x}.$$

$$= c[6x - 6x^2 + x^3] e^{-x}$$

$$y_p''' = c[6 - 18x + 9x^2 - x^3] e^{-x}$$

Substitute these expression in (6) and cancel the common factor e^{-x} gives

$$C(6-18x+9x^2-x^3)+3C(6x-6x^2+x^3)+3C(3x^2-x^3)+Cx^3=30$$

Solving $6C=30$ $\quad \underline{\underline{C=5}}$

$$y_p = 5x^3e^{-x}.$$

<u>Step 3</u> $\quad y = y_h + y_p$

$$= (C_1+C_2x+C_3x^2)e^{-x} + 5x^3e^{-x}$$

$y(0)=3 \implies C_1=3$

$$y' = \left[-3+C_2+(-C_2+2C_3)x+(15-C_3)x^2-5x^3\right]e^{-x}$$

$y'(0)=-3 \implies C_2=0$

$$y'' = \left[3+2C_3+(30-4C_3)x+(-30+C_3)x^2+5x^3\right]e^{-x}$$

$y''(0)=-47 \implies \underline{\underline{C_3=-25}}$

$$\underline{\underline{y=(3-25x^2)e^{-x}+5x^3e^{-x}}}$$

<u>Variation of parameters</u>

Consider the eqn $\quad y''+p(x)y'+q(x)y=r(x)$

Solve the homogeneous part we get

$$y_h = C_1y_1 + C_2y_2.$$

then $\quad y_p = -y_1\int \frac{y_2 r}{W}\,dx + y_2\int \frac{y_1 r}{W}\,dx.$

Where $\quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

Note: Given ODE is written in Standard form.

Pbm 1) Solve $y'' + 4 = \sec x$

$\rightarrow$ A.E $\lambda^2 + 1 = 0$ $\quad \lambda = \pm i$

$$y_h = c_1 \cos x + c_2 \sin x.$$

$y_1 = \cos x \qquad y_2 = \sin x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$y_p = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx$$

$$= -\cos x \int \sin x \times \sec x \cdot dx + \sin x \int \cos x \cdot \sec x \, dx$$

$$= \cos x \ln|\cos x| + x \sin x$$

$$y = y_h + y_p = c_1 \cos x + c_2 \sin x + \cos x \ln|\cos x| + x \sin x$$

2) Solve $y'' - 6y' + 9 = \dfrac{e^{3x}}{x^2}$

$\rightarrow$ A.E $\lambda^2 - 6\lambda + 9 = 0$ $\quad \lambda = 3, 3.$ $\quad y_h = c_1 e^{3x} + c_2 x e^{3x}.$

$y_1 = e^{3x} \qquad y_2 = x e^{3x}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = e^{6x}.$$

$$-y_1 \int \frac{y_2}{W} r \, dx = -e^{3x} \int \frac{x e^{3x}}{e^{6x}} \cdot \frac{e^{3x}}{x^2} dx = -e^{3x} \log x$$

$$y_2 \int \frac{y_1}{W} r(x) dx = x e^{3x} \int \frac{e^{3x}}{e^{6x}} \cdot \frac{e^{3x}}{x^2} dx = -e^{3x}$$

$$y_p = -e^{3x} \log x - e^{3x}.$$

$$y = y_h + y_p = (c_1 + c_2 x) e^{3x} - (\log x + 1) e^{3x}.$$

Module IV – Laplace Transforms.

Defn: If $f(t)$ is a fn defined for all $t \geq 0$, then its Laplace transform, $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = F(s)$.

i.e., $F(s) = \int_0^\infty k(s,t) f(t)\,dt$, where kernel $k(s,t) = e^{-st}$.

The given fn $f(t)$ is called the inverse transform of $F(s)$ and is denoted by $L^{-1}(F)$.

i.e., $L^{-1}(F) = f(t)$.

Ex. $f(t) = 1$, $t \geq 0$.

$$L(1) = \int_0^\infty e^{-st}.1\,dt = \left.\frac{e^{-st}}{-s}\right)_0^\infty = 1/s \quad ;\ s>0.$$

Ex. $f(t) = e^{at}$; $t \geq 0$ and a is a constant.

$$L(e^{at}) = \int_0^\infty e^{-st}.e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \left.\frac{e^{-(s-a)t}}{-(s-a)}\right)_0^\infty = \frac{1}{s-a} ;\ s>a$$

$L(e^{-at}) = \frac{1}{s+a}$

Linearity property

$$L[a f(t) + b g(t)] = a L[f(t)] + b L[g(t)]$$

Pf: By defn, $L[a f(t) + b g(t)] = \int_0^\infty e^{-st}(a f(t) + b g(t))\,dt$

$$= a\int_0^\infty e^{-st} f(t)\,dt + b\int_0^\infty e^{-st} g(t)\,dt$$

$$= a L[f(t)] + b L[g(t)].$$

Ex. $f(t) = \cosh at$ & $\sinh at$

$\cosh at = \frac{e^{at}+e^{-at}}{2}$; $\sinh at = \frac{e^{at}-e^{-at}}{2}$

$$\therefore L(\cosh at) = \frac{1}{2}\left[L(e^{at}) + L(e^{-at})\right] = \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{2s}{s^2-a^2}\right]$$

$$= \frac{s}{s^2-a^2}$$

$$L(\sinh at) = \frac{1}{2}\left[L(e^{at}) - L(e^{-at})\right] = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2}\cdot\frac{2a}{s^2-a^2}$$

$$= \frac{a}{s^2-a^2}$$

$L(e^{iat}) = \frac{1}{s-ia} = \frac{s+ia}{s^2+a^2}$

$L(\cos at + i\sin at) = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$

$\therefore L(\cos at) = \frac{s}{s^2+a^2}$; $L(\sin at) = \frac{a}{s^2+a^2}$

Ex. $f(t) = \cos at$ & $\sin at$.

Let $L_c = L(\cos at)$; $L_s = L(\sin at)$

then $L_c = \int_0^\infty e^{-st}\cos at\,dt = \left.\cos at.\frac{e^{-st}}{-s}\right)_0^\infty - \int_0^\infty -\sin at.a.\frac{e^{-st}}{-s}\,dt$

$$= \frac{1}{s} - \frac{a}{s}\int_0^\infty e^{-st}\sin at\,dt = \frac{1}{s} - \frac{a}{s}L_s.$$

$$L_s = \int_0^\infty e^{-st}\sin at\,dt = \left.\sin at.\frac{e^{-st}}{-s}\right)_0^\infty - \int_0^\infty \cos at.a\frac{e^{-st}}{-s}\,dt$$

$$= \frac{a}{s}L_c.$$

$\therefore L_c = \frac{1}{s} - \frac{a}{s}\left(\frac{a}{s} L_c\right)$

$\therefore L_c \left(1 + \frac{a^2}{s^2}\right) = \frac{1}{s}$ i.e., $L_c \left(\frac{a^2+s^2}{s^2}\right) = \frac{1}{s}$.

$\therefore L_c = \frac{s}{s^2+a^2}$.

$L_s = \frac{a}{s} \cdot L_c = \frac{a}{s} \cdot \frac{s}{s^2+a^2} = \frac{a}{s^2+a^2}$

Ex. $f(t) = t^n$

$L(t^n) = \int_0^\infty e^{-st} \cdot t^n \, dt$ put $st = u$, $s\,dt = du$, $u : 0$ to ∞.

$= \int_0^\infty e^{-u} \cdot \left(\frac{u}{s}\right)^n \cdot \frac{du}{s}$

$= \frac{1}{s^{n+1}} \int_0^\infty e^{-u} u^{n+1-1} du = \frac{1}{s^{n+1}} \cdot \Gamma(n+1)$ Since $\Gamma p = \int_0^\infty e^{-x} x^{p-1} dx$

$= \frac{n!}{s^{n+1}}$ when n is a +ve integer.

Table

$f(t)$	$L[f(t)]$	$f(t)$	$L(f(t))$
1	$1/s$	e^{at}	$1/s-a$
t	$1/s^2$	e^{-at}	$1/s+a$
t^2	$2!/s^3$	$\cos at$	s/s^2+a^2
t^n	$n!/s^{n+1}$	$\sin at$	a/s^2+a^2
		$\cosh at$	s/s^2-a^2
t^a (a +ve)	$\Gamma a+1/s^{a+1}$	$\sinh at$	a/s^2-a^2

First Shifting Theorem.

If $L[f(t)] = F(s)$, then $L[e^{at} f(t)] = F(s-a)$

& $L[e^{-at} f(t)] = F(s+a)$.

Proof. $L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\, dt$.

$\therefore L[e^{at} f(t)] = \int_0^\infty e^{-st} \cdot e^{at} f(t)\, dt = \int_0^\infty e^{-(s-a)t} f(t)\, dt$

$= F(s-a)$.

$L[e^{-at} f(t)] = \int_0^\infty e^{-st} \cdot e^{-at} f(t)\, dt = \int_0^\infty e^{-(s+a)t} f(t)\, dt = F(s+a)$.

Qn. Find $L[\cosh at \cos \omega t]$

Ans. $L[\cosh at \cos \omega t] = L\left[\left(\frac{e^{at}+e^{-at}}{2}\right)\cos \omega t\right] = \frac{1}{2}\left[L(e^{at}\cos \omega t) + L(e^{-at}\cos \omega t)\right]$

$= \frac{1}{2}\left[\frac{s-a}{(s-a)^2+\omega^2} + \frac{s+a}{(s+a)^2+\omega^2}\right]$.

Existence theorem for Laplace Transforms.

If $f(t)$ is defined and piecewise continuous on every finite interval of $t \geq 0$ and satisfies $|f(t)| \leq Me^{kt}$ for all $t \geq 0$ and some constants M and k, then the Laplace transform $L[f(t)]$ exists for all $s > k$.

Q1. Find the Laplace Transforms of (i) $\sin^2 t$ (ii) $\cos^2 t$
(iii) $\sin t \cos 2t$ (iv) $t^2 \cos at$ (v) $\cos\sqrt{t}$ (vi) $\cos(at+b)$
(vii) $\cos(\omega t+\theta)$

Ans. i) $L(\sin^2 t) = L\left(\frac{1-\cos 2t}{2}\right) = \frac{1}{2}\left[L(1) - L(\cos 2t)\right] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$

$$= \frac{2}{s(s^2+4)}.$$

(ii) $L(\cos^2 t) = L\left(\frac{1+\cos 2t}{2}\right) = \frac{1}{2}\left[L(1) + L(\cos 2t)\right] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]$

$$= \frac{1}{2}\left[\frac{s^2+4+s^2}{s(s^2+4)}\right] = \frac{2s^2+4}{2s(s^2+4)} = \frac{s^2+2}{s(s^2+4)}.$$

(iii) $L(\sin t \cos 2t) = L\left(\frac{\sin 3t - \sin t}{2}\right) = \frac{1}{2}\left[\frac{3}{s^2+9} - \frac{1}{s^2+1}\right]$

$$= \frac{1}{2}\left[\frac{3s^2+3-s^2-9}{(s^2+1)(s^2+9)}\right] = \frac{1}{2}\left(\frac{2s^2-6}{(s^2+1)(s^2+9)}\right) = \frac{s^2-3}{(s^2+1)(s^2+9)}$$

(iv) $L(t^2) = 2!/s^3 = 2/s^3$

$$\therefore L(t^2 e^{iat}) = 2/(s-ia)^3 = \frac{2(s+ia)^3}{(s^2+a^2)^3} = \frac{2(s^3+3s^2ia-3sa^2-ia^3)}{(s^2+a^2)^3}.$$

Equating the real part,
$$L(t^2\cos at) = \frac{2(s^3-3a^2s)}{(s^2+a^2)^3} = \frac{2s(s^2-3a^2)}{(s^2+a^2)^3}$$

(v) We have $\cos x = 1 - x^2/2! + x^4/4! - \cdots$

$$\therefore \cos\sqrt{t} = 1 - t/2! + t^2/4! - \cdots$$

$$\therefore L[\cos\sqrt{t}] = L(1) - 1/2!\, L(t) + 1/4!\, L(t^2) - \cdots$$
$$= 1/s - \frac{1}{2!}\frac{1}{s^2} + 1/4!\; 2!/s^3 \cdots$$

(vi) $\cos(at+b) = \cos at \cos b - \sin at \sin b$

$$\therefore L(\cos(at+b)) = \cos b.\, L(\cos at) - \sin b\, L(\sin at)$$
$$= \cos b . \frac{s}{s^2+a^2} - \sin b . \frac{a}{s^2+a^2} = \frac{s\cos b - a\sin b}{s^2+a^2}$$

(vii) $\cos(\omega t+\theta) = \cos\omega t \cos\theta - \sin\omega t \sin\theta$.

$$\therefore L(\cos(\omega t+\theta)) = \cos\theta . L(\cos\omega t) - \sin\theta . L(\sin\omega t)$$
$$= \cos\theta . \frac{s}{s^2+\omega^2} - \sin\theta . \frac{\omega}{s^2+\omega^2} = \frac{s\cos\theta - \omega\sin\theta}{s^2+\omega^2}$$

Qn. find $L[f(t)]$, where $f(t) = \begin{cases} t ; & 0<t<1 \\ 1 & t>1 \end{cases}$

Ans. $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 e^{-st}\cdot t\,dt + \int_1^\infty e^{-st}\cdot 1\,dt$

$$= t\cdot\frac{e^{-st}}{-s}\Big)_0^1 - \int_0^1 1\cdot\frac{e^{-st}}{-s}\,dt + \frac{e^{-st}}{-s}\Big)_1^\infty$$

$$= t\cdot\frac{e^{-st}}{-s} - 1\cdot\frac{e^{-st}}{(-s)^2}\Big]_0^1 - \frac{1}{s}(0 - e^{-s})$$

$$= \cancel{\frac{e^{-s}}{-s}} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} + \cancel{\frac{1}{s}e^{-s}} = \underline{\frac{1}{s^2}(1 - e^{-s})}$$

Change of scale property

If $L[f(t)] = F(s)$, then $L[f(at)] = \frac{1}{a}F(s/a)$

Pf. $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$.

$\therefore L[f(at)] = \int_0^\infty e^{-st} f(at)\,dt$ Let $at = u$, $a\,dt = du$, $u: 0$ to ∞

$$= \int_0^\infty e^{-s\,u/a} f(u)\,du/a$$

$$= \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)u} f(u)\,du = \underline{\frac{1}{a}F(s/a)}$$

Qn If $L[f(t)] = \frac{a}{s^2+4}$, find $L[f(3t)]$.

Ans. $L[f(3t)] = \frac{1}{a}F(s/a) = \frac{1}{3}F(s/3) = \frac{1}{3}\cdot\frac{a}{(s/3)^2+4} = \frac{3a}{s^2+36}$

Inverse Laplace Transform

If $L[f(t)] = F(s)$, then $f(t)$ is called the inverse Laplace transform of $F(s)$ and is denoted by $L^{-1}[F(s)]$.

$F(s)$	$L^{-1}(F(s)) = f(t)$
$1/s$	1
$1/s^2$	t
$1/s^{n+1}$	$t^n/n!$
$1/(s-a)$	e^{at}
$1/(s+a)$	e^{-at}
$\frac{s}{s^2+a^2}$	$\cos at$
$\frac{1}{s^2+a^2}$	$\frac{1}{a}\sin at$
$\frac{s}{s^2-a^2}$	$\cosh at$
$\frac{1}{s^2-a^2}$	$\frac{1}{a}\sinh at$

Qn. find the inverse Laplace Transforms of

(i) $\dfrac{s+1}{s^2+s+1}$ (ii) $\dfrac{1}{(s+2)^2}$ (iii) $\dfrac{1}{s}+\dfrac{1}{s+2}+\dfrac{s}{s^2-4}$ (iv) $\dfrac{s-3}{(s-3)^2+9}$

(v) $\dfrac{s+1}{s^2+25}$ (vi) $\dfrac{1}{(s^2+1)(s^2+9)}$ (vii) $\dfrac{4s+12}{s^2+8s+16}$ (viii) $\dfrac{1}{s^2-4s+8}$

(ix) $\dfrac{\pi}{s^2+4s\pi+3\pi^2}$ (x) $\dfrac{1}{(s+9)(s+6)}$

Ans.

(i) $L^{-1}\left(\dfrac{s+1}{(s+1/2)^2+3/4}\right) = L^{-1}\left(\dfrac{s+1/2+1/2}{(s+1/2)^2+(\frac{\sqrt{3}}{2})^2}\right) = L^{-1}\left(\dfrac{s+1/2}{(s+1/2)^2+(\sqrt{3}/2)^2}\right)+1/2\,L^{-1}\left(\dfrac{1}{(s+1/2)^2+(\sqrt3/2)^2}\right)$

$= e^{-1/2 t}\cos\frac{\sqrt3}{2}t + 1/2\cdot\dfrac{1}{\sqrt3/2}\sin\frac{\sqrt3}{2}t\cdot e^{-1/2 t}$

$= e^{-1/2 t}\left[\cos\frac{\sqrt3}{2}t + \frac{1}{\sqrt3}\sin\frac{\sqrt3}{2}t\right]$

(ii) $L^{-1}\left(\dfrac{1}{(s+2)^2}\right) = e^{-2t}\,t$

(iii) $L^{-1}\left(\dfrac{1}{s}+\dfrac{1}{s+2}+\dfrac{s}{s^2-2^2}\right) = 1+e^{-2t}+\cosh 2t$

(iv) $L^{-1}\left(\dfrac{s-3}{(s-3)^2+3^2}\right) = e^{3t}\cos 3t$

(v) $\dfrac{s+1}{s(s+2)} = \dfrac{A}{s}+\dfrac{B}{s+2}$ $\therefore\ s+1 = A(s+2)+Bs.$

$s=0 \Rightarrow 1 = 2A$ $A = 1/2$

$s=-2 \Rightarrow -1=-2B$ $B = 1/2$

$\therefore L^{-1}\left(\dfrac{s+1}{s(s+2)}\right) = 1/2\,L^{-1}(1/s) + 1/2\,L^{-1}(1/(s+2))$

$= 1/2 + 1/2\,e^{-2t}$

(vi) $L^{-1}\left(\dfrac{1}{(s^2+1)(s^2+9)}\right)$: $\dfrac{1}{(s^2+1)(s^2+9)} = \dfrac{A}{s^2+1}+\dfrac{B}{s^2+9}.$

$\therefore\ 1 = A(s^2+9)+B(s^2+1)$

Equating coeffs: $A+B=0.$ $9A+B=1$

$A+B=0$

$8A = 1$ $A = 1/8$ $B = -1/8$

$\therefore L^{-1}\left(\dfrac{1}{(s^2+1)(s^2+9)}\right) = L^{-1}\left(\dfrac{1/8}{s^2+1}+\dfrac{-1/8}{s^2+9}\right) = 1/8\,L^{-1}\left(\dfrac{1}{s^2+1}\right) - 1/8\,L^{-1}\left(\dfrac{1}{s^2+9}\right)$

$= 1/8\cdot\sin t - 1/8\cdot\frac{1}{3}\sin 3t = \frac{1}{24}(3\sin t - \sin 3t)$

(vii) $L^{-1}\left(\dfrac{4s+12}{s^2+8s+16}\right) = L^{-1}\left(\dfrac{4(s+4)-4}{(s+4)^2}\right) = 4\,L^{-1}\left(\dfrac{s+4}{(s+4)^2}\right) - 4\,L^{-1}\left(\dfrac{1}{(s+4)^2}\right)$

$= 4.\,L^{-1}\left(\dfrac{1}{s+4}\right) - 4\,L^{-1}\left(\dfrac{1}{(s+4)^2}\right)$

$= 4e^{-4t} - 4e^{-4t}t = 4e^{-4t}(1-t)$

viii) $L^{-1}\left(\frac{1}{s^2-4s+8}\right) = L^{-1}\left(\frac{1}{(s-2)^2+2^2}\right) = \frac{1}{2}\sin 2t \cdot e^{2t} = \frac{e^{2t}\sin 2t}{2}$

ix) $L^{-1}\left(\frac{\lambda}{s^2+4s\lambda+3\lambda^2}\right) = L^{-1}\left(\frac{\lambda}{(s+2\lambda)^2-\lambda^2}\right) = \lambda \cdot L^{-1}\left(\frac{1}{(s+2\lambda)^2-\lambda^2}\right)$

$= \lambda \cdot \frac{1}{\lambda}\sinh \lambda t \cdot e^{-2\lambda t} = \underline{e^{-2\lambda t}\sinh \lambda t}$

(x)

$\frac{1}{(s+a)(s+b)} = \frac{A}{s+a} + \frac{B}{s+b}$ $\quad \therefore 1 = A(s+b) + B(s+a)$

$s=-a \Rightarrow 1 = A(b-a) \quad \therefore A = \frac{-1}{a-b}$

$s=-b \Rightarrow 1 = B(-b+a) \quad B = 1/_{a-b}$

$\therefore L^{-1}\left(\frac{1}{(s+a)(s+b)}\right) = L^{-1}\left(\frac{-1/_{a-b}}{s+a} + \frac{1/_{a-b}}{s+b}\right) = \frac{-1}{(a-b)}e^{-at} + \frac{1}{(a-b)}e^{-bt}$

$= \underline{\frac{1}{(a-b)}\left[-e^{-at} + e^{-bt}\right]}$

<u>Laplace Transform of derivatives</u>

$L(y') = sL(y) - y(0)$

$L(y'') = s^2L(y) - sy(0) - y'(0)$

$L(y''') = s^3L(y) - s^2y(0) - sy'(0) - y''(0)$

<u>Differentiation</u> of Transforms

If $L[f(t)] = F(s)$, then $L(t^n f(t)) = (-1)^n \frac{d^n}{ds^n}F(s)\ ;\ n=1,2,\ldots$

<u>Laplace Transform of Integral</u>

If $L[f(t)] = F(s)$, then $L\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}F(s)$

$\therefore L^{-1}\left[\frac{1}{s}F(s)\right] = \int_0^t f(\tau)\,d\tau$

<u>Q</u> Find the Laplace transforms of (i) $t^3 e^{-3t}$ (ii) $t\sinh at$ (iii) $t\cosh at$

<u>Ans.</u> $L(t^3) = 3!/s^4$

$\therefore L(e^{-3t}t^3) = 3!/(s+3)^4 = 6/(s+3)^4$

<u>OR</u> $L(e^{-3t}) = \frac{1}{s+3}$

$\therefore L(t^3e^{-3t}) = (-1)^3\frac{d^3}{ds^3}\left(\frac{1}{s+3}\right)$

$= -\frac{d^2}{ds^2}\left(\frac{-1}{(s+3)^2}\right) = \frac{d}{ds}\left[\frac{-2}{(s+3)^3}\right]$

$= \underline{6/(s+3)^4}$

(ii) $L(\sinh at) = \frac{a}{s^2-a^2}$. $\quad \therefore L(t\sinh at) = (-1)^1\frac{d}{ds}\left(\frac{a}{s^2-a^2}\right)$

$= -a\cdot\frac{-1\cdot 2s}{(s^2-a^2)^2} = \frac{2as}{(s^2-a^2)^2}$.

(iii) $L(\cosh at) = \frac{s}{s^2-a^2}$ $\quad \therefore L(t\cosh at) = (-1)^1\frac{d}{ds}\left[\frac{s}{s^2-a^2}\right] = -\left[\frac{(s^2-a^2)-s\cdot 2s}{(s^2-a^2)^2}\right]$

$$= \frac{s^2+a^2}{(s^2-a^2)^2}$$

Integration of Transforms

If $L[f(t)] = F(s)$, then $L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds.$

Q. Find $L\left[\frac{\cos 2t - \cos 3t}{t}\right]$

Ans. we have $L[\cos 2t] = \frac{s}{s^2+4}$; $L(\cos 3t) = \frac{s}{s^2+9}.$

$$\therefore L\left[\frac{\cos 2t - \cos 3t}{t}\right] = \int_s^\infty \left(\frac{s}{s^2+4} - \frac{s}{s^2+9}\right) ds$$

$$= \frac{1}{2}\int_s^\infty \frac{2s}{s^2+4} - \frac{2s}{s^2+9}\,ds = \frac{1}{2}\left[\log(s^2+4) - \log(s^2+9)\right]_s^\infty$$

$$= \frac{1}{2}\log\frac{s^2+4}{s^2+9}\Big]_s^\infty = \frac{1}{2}\log\frac{1+4/s^2}{1+9/s^2}\Big]_s^\infty = -\frac{1}{2}\left[\log 1 - \log\left(\frac{s^2+4}{s^2+9}\right)\right]$$

$$= \frac{1}{2}\log\frac{s^2+9}{s^2+4}$$

Solutions to ODE using Laplace Transform

$L(y') = sL(y) - y(0)$

$L(y'') = s^2L(y) - s\,y(0) - y'(0)$

$L(y''') = s^3L(y) - s^2y(0) - s\,y'(0) - y''(0)$

Q. Using Laplace transform solve $y' + 4y = t$, $y(0) = 1$.

Ans. $L(y') + 4L(y) = L(t)$ Let $L(y) = Y$

$sL(y) - y(0) + 4L(y) = 1/s^2$

i.e., $sY - 1 + 4Y = \frac{1}{s^2}$

i.e., $(s+4)Y = 1 + \frac{1}{s^2} = \frac{s^2+1}{s^2}$

$$\therefore Y = \frac{s^2+1}{s^2(s+4)}$$

Now $\frac{s^2+1}{s^2(s+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+4}.$

$\therefore s^2+1 = As(s+4) + B(s+4) + Cs^2$

$s=0 \Rightarrow 1 = 4B$ $\quad B = 1/4$

$s=-4 \Rightarrow 17 = 16C$ $\quad C = 17/16$

Now $A + C = 1$ $\quad \therefore A = 1 - 17/16 = -1/16$

$$\therefore Y = \frac{-1/16}{s} + \frac{1/4}{s^2} + \frac{17/16}{s+4}.$$

i.e., $L(y) = -1/16\,\frac{1}{s} + \frac{1}{4}\cdot\frac{1}{s^2} + \frac{17}{16}\cdot\frac{1}{s+4}$

$\therefore y = -1/16\,L^{-1}(1/s) + \frac{1}{4}L^{-1}(1/s^2) + 17/16\,L^{-1}\left(\frac{1}{s+4}\right)$

$= -1/16 + 1/4\,t + 17/16\,e^{-4t} = 1/16\left[17e^{-4t} + 4t - 1\right]$

Q. Using Laplace Transform solve the IVP $y'' - 3y' + 2y = 4e^{2t}$, $y(0) = -3$, $y'(0) = 5$

An. $L(y'') - 3L(y') + 2L(y) = 4L(e^{2t})$.

$s^2 L(y) - s\,y(0) - y'(0) - 3[sL(y) - y(0)] + 2L(y) = 4 \cdot \frac{1}{s-2}$.

Let $L(y) = Y$.

$s^2 Y + 3s \cancel{-5} - \cancel{3sY} - 9 + 2Y = 4 \frac{1}{s-2}$ (-5, $-3sY$)

$\therefore (s^2 - 3s + 2) Y = \frac{4}{s-2} - 3s + 5 + 9 = \frac{4 - 3s^2 + 6s + 5s - 10 + 9s - 18}{(s-2)}$

$\therefore Y = \frac{-3s^2 + 20s - 24}{(s-2)(s-2)(s-1)} = \frac{-3s^2 + 20s - 24}{(s-1)(s-2)^2}$

Now $\frac{-3s^2 + 20s - 24}{(s-1)(s-2)^2} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$

$\therefore -3s^2 + 20s - 24 = A(s-2)^2 + B(s-1)(s-2) + C(s-1)$

$s = 1 \Rightarrow -7 = A$ $\qquad -3 = A + B \qquad B = -3 + 7 = 4$

$s = 2 \Rightarrow 4 = C$

$\therefore Y = \frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$

$\therefore L(y) = -7 \cdot \frac{1}{s-1} + 4 \cdot \frac{1}{s-2} + 4 \cdot \frac{1}{(s-2)^2}$

$\therefore y = -7 L^{-1}\left(\frac{1}{s-1}\right) + 4 \cdot L^{-1}\left(\frac{1}{s-2}\right) + 4 \cdot L^{-1}\left(\frac{1}{(s-2)^2}\right)$

$\therefore y = -7e^{t} + 4e^{2t} + 4te^{2t}$

Q. Using Laplace Transform, solve $y'' + y' + 9y = 0$, $y(0) = 0.16$, $y'(0) = 0$

An: $L(y'') + L(y') + 9L(y) = L(0)$

$s^2 L(y) - s\,y(0) - y'(0) + [sL(y) - y(0)] + 9L(y) = 0$

Let $L(y) = Y$

$s^2 Y - 0.16s + sY - 0.16 + 9Y = 0$.

$Y(s^2 + s + 9) = 0.16s + 0.16 = 0.16(s+1)$

$\therefore Y = \frac{0.16(s+1)}{s^2 + s + 9} = \frac{0.16(s + \frac{1}{2} + \frac{1}{2})}{(s + \frac{1}{2})^2 + \frac{35}{4}} = \frac{0.16(s + 1/2)}{(s + 1/2)^2 + (\frac{\sqrt{35}}{2})^2} + \frac{0.08}{(s + 1/2)^2 + (\frac{\sqrt{35}}{2})^2}$

$\therefore L(y) = 0.16 \cdot \frac{(s + 1/2)}{(s + 1/2)^2 + (\frac{\sqrt{35}}{2})^2} + 0.08 \cdot \frac{1}{(s + 1/2)^2 + (\frac{\sqrt{35}}{2})^2}$

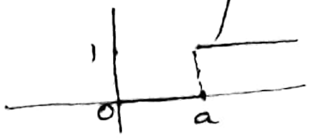
$\therefore y = 0.16\, L^{-1}\left(\frac{s + 1/2}{(s + 1/2)^2 + (\frac{\sqrt{35}}{2})^2}\right) + 0.08 \cdot L^{-1}\left(\frac{1}{(s + 1/2)^2 + (\frac{\sqrt{35}}{2})^2}\right)$

$\therefore y = 0.16\, e^{-1/2 t} \cos\frac{\sqrt{35}}{2}t + 0.08 \frac{1}{\frac{\sqrt{35}}{2}} \sin\frac{\sqrt{35}}{2}t \cdot e^{-1/2 t}$

$\therefore y = 0.16\, e^{-1/2 t} \cos\frac{\sqrt{35}}{2}t + \frac{0.16}{\sqrt{35}} \sin\frac{\sqrt{35}}{2}t \cdot e^{-1/2 t}$

Unit step fn The unit step fn $u(t-a)$ is defined as

$$u(t-a) = \begin{cases} 0 & \text{for } t<a \\ 1 & \text{for } t \geq a \end{cases} ; \quad a \geq 0$$



In particular $u(t) = 0$ for $t<0$, 1 for $t \geq 0$

Laplace Transform of unit step fn

$L(u(t-a)) = \int_0^\infty e^{-st} u(t-a)\, dt$. But $u(t-a) = \begin{cases} 0 & \text{for } t<a \\ 1 & \text{for } t \geq a \end{cases}$

$= \int_0^a e^{-st} \cdot 0\, dt + \int_a^\infty e^{-st} \cdot 1\, dt = \left.\frac{e^{-st}}{-s}\right)_a^\infty = -\frac{1}{s}\left(0 - e^{-as}\right)$

$= \frac{e^{-as}}{s}$.

Second Shifting theorem

If $L[f(t)] = F(s)$, then $L[f(t-a)\, u(t-a)] = e^{-as} F(s)$.

Pf. $L[f(t-a)\, u(t-a)] = \int_0^\infty e^{-st} \cdot f(t-a)\, u(t-a)\, dt$

$= \int_a^\infty e^{-st} \cdot f(t-a) \cdot 1\, dt$ Let $t-a = x$, $dt = dx$, $x: 0 \text{ to } \infty$

$= \int_0^\infty e^{-s(a+x)} f(x)\, dx$

$= e^{-as} \int_0^\infty e^{-sx} f(x) dx = e^{-as} F(s)$.

Qn. Find the Laplace Transforms of

(i) $(t-3)^2 u(t-3)$ (ii) $\sin t\, u(t-\pi)$ (iii) $e^{-t}[1 - u(t-2)]$

(iv) $t^4 u_2(t)$

Ans. i) $a = 3$. $f(t) = t^2 \Rightarrow F(s) = 2/s^3$.

$\therefore L[(t-3)^2 u(t-3)] = L[f(t-a)\, u(t-a)] = e^{-as} F(s) = e^{-3s} \cdot 2/s^3$

(ii) $\sin t = \sin(t - \pi + \pi) = -\sin(t-\pi)$.

$\therefore \sin t\, u(t-\pi) = -\sin(t-\pi) \cdot u(t-\pi)$.

$a = \pi$. $f(t) = \sin t \Rightarrow F(s) = \frac{1}{s^2+1}$.

$\therefore L[\sin(t-\pi)\, u(t-\pi)] = -e^{-as} F(s) = -e^{-\pi s} \cdot \frac{1}{s^2+1} = -\frac{e^{-\pi s}}{s^2+1}$

iii) $e^{-t}[1 - u(t-2)] = e^{-t} - e^{-t} u(t-2)$

$= e^{-t} - e^{-(t-2)} \cdot u(t-2) \cdot e^{-2}$.

$\therefore L[e^{-t}(1-u(t-2)] = L(e^{-t}) - e^{-2}\, L(e^{-(t-2)} \cdot u(t-2))$ $\quad a=2$

$f(t)=e^{-t} \Rightarrow F(s)=\frac{1}{s+1}$

$= \frac{1}{s+1} - e^{-2} \cdot e^{-2s} \cdot \frac{1}{s+1}$

$= \frac{1 - e^{-2(s+1)}}{s+1}$

(iv) $t^4 u_2(t) = t^4 u(t-2) = (t-2+2)^4 u(t-2)$

$= [(t-2)^4 + 8(t-2)^3 + 24(t-2)^2 + 32(t-2) + 16]\, u(t-2)$

$= [(t-2)^4 + 8(t-2)^3 + 24(t-2)^2 + 32(t-2)]u(t-2)$

$= (t-2)^4 u(t-2) + 8(t-2)^3 u(t-2) + 24(t-2)^2 u(t-2) + 32(t-2)u(t-2) + 16\, u(t-2)$

$f(t) = t^4 \Rightarrow F(s) = 4!/s^5$

$\therefore L(t^4 u_2(t)) = e^{-2s}\left[4!/s^5 + 8 \cdot 3!/s^4 + 24 \cdot 2!/s^3 + 32 \cdot 1/s^2 + 16\, 1/s\right]$

$= 8 e^{-2s}\left[3/s^5 + 6/s^4 + 6/s^3 + 4/s^2 + 2/s\right]$

$= \frac{8e^{-2s}}{s}\left[3/s^4 + 6/s^3 + 6/s^2 + 4/s + 2\right]$

Model Qn Find the Laplace Transform of $e^{-t}\sinh 4t$.

Ans. $L(\sinh 4t) = \frac{4}{s^2-16}$.

$\therefore L(e^{-t}\sinh 4t) = \frac{4}{(s+1)^2-16} = \frac{4}{s^2+2s-15}$

Qn Find the Laplace inverse Transform of $\frac{1}{s(s^2+\omega^2)}$

Ans. $\frac{1}{s(s^2+\omega^2)} = \frac{A}{s} + \frac{Bs+C}{s^2+\omega^2}$

$\therefore 1 = A(s^2+\omega^2) + (Bs+C)s$

$A+B=0$; $C=0$, $A\omega^2=1$ $\quad A = 1/\omega^2$, $B = -1/\omega^2$

$\therefore \frac{1}{s(s^2+\omega^2)} = \frac{1/\omega^2}{s} + \frac{-1/\omega^2\, s}{s^2+\omega^2}$

$\therefore L^{-1}\left(\frac{1}{s(s^2+\omega^2)}\right) = \frac{1}{\omega^2}\left[L^{-1}(1/s) - L^{-1}\left(\frac{s}{s^2+\omega^2}\right)\right]$

$= \frac{1}{\omega^2}\left[1 - \cos\omega t\right]$

Convolution theorem

If $L^{-1}[F(s)] = f(t)$ and $L^{-1}(G(s)) = g(t)$, then

$L^{-1}[F(s) \cdot G(s)] = \int_0^t f(u)\, g(t-u)\, du$.

Qn. Apply Convolution theorem to evaluate the inverse Laplace transform

(i) $\frac{s}{(s^2+4)^2}$ (ii) $\frac{16}{(s-2)(s+2)^2}$

Ans. (i) $L^{-1}\left[\frac{s}{(s^2+4)^2}\right] = L^{-1}\left[\frac{s}{s^2+4} \cdot \frac{1}{s^2+4}\right]$

Let $F(s) = \frac{s}{s^2+4} \Rightarrow f(t) = \cos 2t$.

$G(s) = \frac{1}{s^2+4} \Rightarrow g(t) = \frac{\sin 2t}{2}$

By Convolution theorem,

$L^{-1}[F(s) \cdot G(s)] = \int_0^t f(u)\, g(t-u)\, du$

$= \int_0^t \cos 2u \cdot \frac{\sin 2(t-u)}{2}\, du$

$= \frac{1}{4}\int_0^t 2\cos 2u \sin(2t-2u)\, du$

$= \frac{1}{4}\int_0^t \sin 2t - \sin(4u-2t)\, du = \frac{1}{4}\left[u \sin 2t + \frac{\cos(4u-2t)}{4}\right]_0^t$

$= \frac{1}{4}\left[t \sin 2t + \frac{\cos 2t}{4} - \left(0 + \frac{\cos(-2t)}{4}\right)\right]$

$= \frac{1}{4}\left[t \sin 2t + \frac{\cos 2t}{4} - \frac{\cos 2t}{4}\right] = \frac{1}{4} t \sin 2t$

(ii) $L^{-1}\left[\frac{16}{(s-2)(s+2)^2}\right] = 16 \cdot L^{-1}\left[\frac{1}{s-2} \cdot \frac{1}{(s+2)^2}\right]$

Let $F(s) = \frac{1}{s-2} \Rightarrow f(t) = e^{2t}$

$G(s) = \frac{1}{(s+2)^2} \Rightarrow g(t) = e^{-2t}\, t$

By Convolution theorem, $L^{-1}[F(s) \cdot G(s)] = \int_0^t f(u)\, g(t-u)\, du$

$= \int_0^t e^{2u} \cdot e^{-2(t-u)} \cdot (t-u)\, du = \int_0^t e^{4u-2t}(t-u)\, du$

$= e^{-2t}\int_0^t e^{4u} \cdot (t-u)\, du = e^{-2t}\left[(t-u)\frac{e^{4u}}{4} - (-1)\frac{e^{4u}}{16}\right]_0^t$

$= e^{-2t}\left[\frac{e^{4t}}{16} - \left\{\frac{t}{4} + \frac{1}{16}\right\}\right] = \frac{e^{+2t}}{16} - \frac{t e^{-2t}}{4} - \frac{e^{-2t}}{16}$

$\therefore L^{-1}\left[\frac{16}{(s-2)(s+2)^2}\right] = e^{2t} - 4te^{-2t} - e^{-2t}$

Qn. Find $L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{s}{s^2+a^2} \cdot \frac{s}{s^2+b^2}\right]$

Let $F(s) = \frac{s}{s^2+a^2} \Rightarrow f(t) = \cos at$

$$G(s) = \frac{s}{s^2+b^2} \rightarrow g(t) = \cos bt$$

$$L^{-1}[F(s)G(s)] = \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t \cos au \; \cos b(t-u)\, du = \frac{1}{2}\int_0^t 2\cos au \cdot \cos(bt-bu)\, du$$

$$= \frac{1}{2}\int_0^t \cos(au+bt-bu) + \cos(au-bt+bu)\, du$$

$$= \frac{1}{2}\left[\frac{\sin(au-bu+bt)}{(a-b)} + \frac{\sin(au+bu-bt)}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\left\{\frac{\sin(at-\cancel{bt}+\cancel{bt})}{a-b} + \frac{\sin(at+\cancel{bt}-\cancel{bt})}{a+b}\right\} - \left\{\frac{\sin bt}{a-b} + \frac{\sin(-bt)}{a+b}\right\}\right]$$

$$= \frac{1}{2}\left[\frac{\sin at}{a-b} + \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right]$$

$$= \frac{1}{2}\left[\sin at\left(\frac{a+\cancel{b}+a-\cancel{b}}{a^2-b^2}\right) + \sin bt\left(\frac{-\cancel{a}-b+\cancel{a}-b}{a^2-b^2}\right)\right]$$

$$= \frac{1}{\cancel{2}}\left[\frac{\cancel{2}a}{a^2-b^2}\sin at - \frac{\cancel{2}b}{a^2-b^2}\sin bt\right] = \frac{a\sin at - b\sin bt}{a^2-b^2}$$

Qn. Find $L^{-1}\left(\frac{1}{(s^2+a^2)^2}\right)$ using convolution theorem.

Let $F(s) = \frac{1}{s^2+a^2} \Rightarrow f(t) = \frac{\sin at}{a}$

$G(s) = \frac{1}{s^2+a^2} \rightarrow g(t) = \frac{\sin at}{a}$.

By convolution theorem,

$$L^{-1}[F(s)G(s)] = \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t \frac{\sin au}{a} \cdot \frac{\sin a(t-u)}{a}\, du = \frac{-1}{2a^2}\int_0^t -2\sin au \; \sin(at-au)\, du$$

$$= -\frac{1}{2a^2}\int_0^t \cos(at) - \cos(2au-at)\, du$$

$$= -\frac{1}{2a^2}\left[u\cos at - \frac{\sin(2au-at)}{2a}\right]_0^t$$

$$= \frac{-1}{2a^2}\left[t\cos at - \frac{\sin at}{2a} - \left(0 - \frac{\sin(-at)}{2a}\right)\right]$$

$$= -\frac{1}{2a^2}\left[t\cos at - \frac{\sin at}{2a} - \frac{\sin at}{2a}\right] = \frac{-1}{2a^2}\left[t\cos at - \frac{\sin at}{a}\right]$$

$$= \frac{1}{2a^2}\left[\frac{\sin at}{a} - t\cos at\right]$$

Dirac Delta fn and its Laplace Transform

The dirac delta fn is defined as

$$\delta(t-a) = \begin{cases} \infty & \text{if } t=0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \int_0^\infty \delta(t-a)\,dt = 1.$$

$$L[\delta(t-a)] = \int_0^\infty e^{-st}\,\delta(t-a)\,dt$$

But $\int_{-\infty}^{\infty} f(t)\,\delta(t-a)\,dt = f(a)$

$\therefore \int_0^\infty e^{-st}\,\delta(t-a)\,dt = e^{-sa}$

i.e, $L[\delta(t-a)] = e^{-as}$.

Solution of ODE involving unit step fn and Dirac delta fn

Qn. Solve the ODE $y'' + 3y' + 2y = \delta(t-1)$; $y(0)=0$, $y'(0)=0$.

Ans. $L(y'') + 3L(y') + 2L(y) = L[\delta(t-1)]$

i.e, $s^2L(y) - sy(0) - y'(0) + 3[sL(y) - y(0)] + 2L(y) = e^{-s}$

let $L(y) = Y$

$s^2Y + 3sY + 2Y = e^{-s}$

i.e, $(s^2+3s+2)Y = e^{-s}$

$$Y = \frac{e^{-s}}{s^2+3s+2} = \frac{e^{-s}}{(s+1)(s+2)} = \left(\frac{1}{s+1} - \frac{1}{s+2}\right)e^{-s}$$

i.e, $L(y) = \left(\frac{1}{s+1} - \frac{1}{s+2}\right)e^{-s} = \frac{1}{s+1}e^{-s} - \frac{1}{s+2}e^{-s}$

$\therefore y = L^{-1}\left(\frac{1}{s+1}e^{-s}\right) - L^{-1}\left(\frac{1}{s+2}e^{-s}\right)$

$L^{-1}\left(\frac{1}{s+1}\right) = e^{-t} = f(t)$; $L^{-1}\left(\frac{1}{s+2}\right) = e^{-2t} = g(t)$

$\therefore L^{-1}\left(\frac{1}{s+1}e^{-s}\right) = f(t-1)\,u(t-1) = e^{-(t-1)}u(t-1)$

$L^{-1}\left(\frac{1}{s+2}e^{-s}\right) = f(t-1)\,u(t-1) = e^{-2(t-1)}u(t-1)$

$$\therefore y = e^{-(t-1)}u(t-1) - e^{-2(t-1)}u(t-1) = \left[e^{-(t-1)} - e^{-2(t-1)}\right]u(t-1)$$

Qn. Find i) $L^{-1}\left(\frac{s\,e^{-3s}}{s^2-9}\right)$ (ii) $L^{-1}\left(\frac{e^{-2s}}{s+3}\right)$ (iii) $L^{-1}\left(\frac{e^{-2s}}{s^2}\right)$

iv) $L^{-1}\left[\frac{(3s+1)\,e^{-3s}}{s^2(s^2+4)}\right]$

An (i) $L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at = f(t)$.

$\therefore L^{-1}\left[F(s)\,e^{-as}\right] = f(t-a)\,U(t-a)$; $a = 3$

$\therefore L^{-1}\left(e^{-3s}\cdot\frac{s}{s^2-a^2}\right) = \cosh a(t-3)\,U(t-3)$

(ii) $L^{-1}\left(\frac{1}{s+3}\right) = e^{-3t} = f(t)$; $a = 2$

$L^{-1}\left(e^{-2s}\cdot\frac{1}{s+3}\right) = f(t-a)\,U(t-a)$
$= e^{-3(t-2)}\,U(t-2)$

(iii) $L^{-1}\left(\frac{1}{s^2}\right) = t = f(t)$ $a = 2$

$\therefore L^{-1}\left(e^{-2s}\cdot\frac{1}{s^2}\right) = f(t-2)\,U(t-2)$
$= (t-2)\,U(t-2)$

(iv) $\frac{3s+1}{s^2(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4}$.

$B = 1/4$ $A = 3/4$, $C = -3/4$, $D = -1/4$

$F(s) = \frac{3}{4}\cdot 1/s + 1/4\cdot\frac{1}{s^2} + \frac{-3/4 s + -1/4}{s^2+4}$

$\therefore f(t) = 3/4 + 1/4\, t - 3/4\cos 2t - 1/4\,\frac{\sin 2t}{2}$

$L^{-1}\left(e^{-3s}F(s)\right) = f(t-3)\,U(t-3)$
$= \left[3/4 + 1/4(t-3) - 3/4\cos 2(t-3) - 1/8\sin 2(t-3)\right]U(t-3)$

Qn Determine the response of the damped mass-spring system under a square wave, modeled by
$y'' + 3y' + 2y = u(t-1) - u(t-2)$; $y(0) = 0$, $y'(0) = 0$.

An. $L(y'') + 3L(y') + 2L(y) = L(U(t-1)) - L(U(t-2))$

$s^2L(y) - sy(0) - y'(0) + 3\left[sL(y) - y(0)\right] + 2L(y) = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}$

$\therefore s^2Y + 3sY + 2Y = \frac{e^{-s} - e^{-2s}}{s}$

$Y = \frac{1}{s^2+3s+2}\;\frac{e^{-s} - e^{-2s}}{s}$

$\therefore L(y) = \frac{1}{s(s+1)(s+2)}\left(e^{-s} - e^{-2s}\right)$

$F(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$. $1 = A(s+1)(s+2) + Bs(s+2) + Cs(s+1)$

$s=0 \Rightarrow 1 = 2A$ $A = 1/2$; $s=-1 \Rightarrow 1 = -B$ $s=-2 \Rightarrow 1 = 2C$

$\because F(s) = \frac{1/2}{s} - \frac{1}{s+1} + \frac{1/2}{s+2} \Rightarrow f(t) = 1/2 - e^{-t} + 1/2\, e^{-2t}$.

$\therefore y = L^{-1}\left[F(s).\, e^{-s}\right] - L^{-1}\left\{F(s)\, e^{-2s}\right\}$

$= f(t-1)\, U(t-1) - f(t-2)\, U(t-2)$

$= \left[1/2 - e^{-(t-1)} + 1/2\, e^{-2(t-1)}\right] U(t-1) - \left[1/2 - e^{-(t-2)} + 1/2\, e^{-2(t-2)}\right] U(t-2)$

Qn. Find i) $L(t e^{2t} \sin 3t)$ (ii) $L(t \sin 3t \cos 2t)$ (iii) $L(t^2 e^{-2t} \cos t)$

Ans i) $L(\sin 3t) = \frac{3}{s^2+9}$

$L(t \sin 3t) = -\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = \frac{6s}{(s^2+9)^2}$

$\therefore L(e^{2t}.\, t \sin 3t) = \frac{6(s-2)}{\left[(s-2)^2+9\right]^2} = \frac{6s-12}{(s^2-4s+13)^2}$

(ii) $L(\sin 3t \cos 2t] = \frac{1}{2} L[\sin 5t + \sin t] = \frac{1}{2}\left[\frac{5}{s^2+25} + \frac{1}{s^2+1}\right]$

$\therefore L[t \sin 3t \cos 2t] = -\frac{d}{ds}\left[1/2\left(\frac{5}{s^2+25} + \frac{1}{s^2+1}\right)\right]$

$= \frac{5s}{(s^2+25)^2} + \frac{s}{(s^2+1)^2}$

(iii) $L(\cos t) = \frac{s}{s^2+1}$

$L(t^2 \cos t) = (-1)^2 \frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right) = \frac{2s^3 - 6s}{(s^2+1)^3}$

$\therefore L(e^{-2t}\, t^2 \cos t) = \frac{2(s+2)^3 - 6(s+2)}{\left[(s+2)^2+1\right]^3} = \frac{2s^3 + 12s^2 + 18s + 4}{(s^2+4s+5)^3}$

Qn. Using Laplace Transform, evaluate

$\int_0^\infty t \sin t\, e^{-t}\, dt$.

Ans. $\int_0^\infty e^{-t}(t \sin t)\, dt = L(t \sin t)$, when $s = 1$

But $L(\sin t) = \frac{1}{s^2+1}$.

$\therefore L(t \sin t) = (-1)^1 \frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2} = \frac{2 \times 1}{(1+1)^2} = 1/2$

$\therefore \int_0^\infty t \sin t\, e^{-t}\, dt = 1/2$

Qn Find the Laplace Transform of $f(t)$ defined by

$f(t) = \begin{cases} t/T & ; 0 < t < T \\ 1 & t > T. \end{cases}$

Ans. $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = \int_0^T e^{-st}.\, \frac{t}{T}\, dt + \int_T^\infty e^{-st}.\, 1\, dt$

$= \frac{1}{T}\left[t.\, \frac{e^{-st}}{-s} - (1).\, \frac{e^{-st}}{(-s)^2}\right]_0^T + \left(\frac{e^{-st}}{-s}\right)_T^\infty = \frac{1 - e^{-sT}}{Ts^2}$ //.

Qn. Find the Laplace Transform of $f(t) = \begin{cases} t^2 ; & 0<t<2 \\ t-1 , & 2<t<3 \\ 7 ; & t>3 \end{cases}$

Ans $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$= \int_0^2 e^{-st} \cdot t^2\,dt + \int_2^3 e^{-st}(t-1)\,dt + \int_3^\infty e^{-st}\, 7\,dt$

$= \left[t^2 \cdot \frac{e^{-st}}{-s} - (2t)\frac{e^{-st}}{(-s)^2} + 2\frac{e^{-st}}{(-s)^3} \right]_0^2 + \left[(t-1)\frac{e^{-st}}{-s} - (1)\frac{e^{-st}}{(-s)^2} \right]_2^3$

$+ 7\left(\frac{e^{-st}}{-s} \right)_3^\infty = 2/s^3 - \frac{e^{-2s}}{s^3}(2+3s+3s^2) + \frac{e^{-3s}}{s^2}(5s-1)$

Qn. Evaluate $L^{-1}\left(\frac{1}{s(s^2-a^2)} \right)$

Ans. We have $L^{-1}\left[\frac{F(s)}{s} \right] = \int_0^t f(\tau)\,d\tau)$

Here $F(s) = 1/s^2-a^2 \Rightarrow f(t) = \frac{\sinh at}{a}$

$\therefore L^{-1}\left(\frac{1/s^2-a^2}{s} \right) = \int_0^t \frac{\sinh a\tau}{a}\,d\tau = \frac{1}{a} \cdot \left. \frac{\cosh a\tau}{a} \right]_0^t = \frac{1}{a^2}\left[\cosh at - 1\right]$

Qn. Find $L^{-1}\left[\tan^{-1}(2/s) \right]$.

Let $F(s) = \tan^{-1}(2/s) \Rightarrow F'(s) = \frac{1}{1+(2/s)^2} \cdot -2/s^2 = \frac{-2}{s^2+4}$.

we have $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$

$\therefore L[t\, f(t)] = -\frac{d}{ds} F(s) = -F'(s)$ $\therefore$ ~~$L^{-1}(F'(s)) = -t f(t)$~~.

$\therefore t\, f(t) = -L^{-1}(F'(s))$

$= -L^{-1}\left(\frac{-2}{s^2+4} \right) = \sin 2t$.

$\therefore f(t) = \frac{\sin 2t}{t}$.

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MODULE – III (1)

Mod 1

FOURIER INTEGRALS AND TRANSFORM

Periodic fns:: A function $f(x)$ is called periodic if $f(x+p) = f(x)$, for all x. The smallest positive value of p is called period.

$\sin(x+2\pi) = \sin(x+4\pi) = \dots = \sin x$. $\therefore \sin x$ is periodic with period 2π.

The Fourier Series of $f(x)$ is $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

in the interval $(-\pi, \pi)$. a_n & b_n are called Fourier Coefficients.

Fourier Integral representation – The Fourier integral representation of a real valued fn $f(x)$ is given by

$$f(x) = \int_0^{\infty}\left(A(\omega)\cos\omega x + B(\omega)\sin\omega x\right)d\omega \text{, where}$$

$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x\,dx$, $B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x\,dx$, where x is a point in which the fn is continuous. If x is a point of discontinuity, then the value of the above integral representation is $\frac{1}{2}\left[f(x^+) + f(x^-)\right]$, where $f(x^+)$ is the RHL and $f(x^-)$ the LHL.

Fourier Integral Theorem: If $f(x)$ is piecewise continuous in every finite interval and has a right hand derivative and a left hand derivative at every point and if the integral $\int_{-\infty}^{\infty} |f(x)|\,dx$ exists, then $f(x)$ can be represented by a Fourier integral, $f(x) = \int_0^{\infty}\left(A(\omega)\cos\omega x + B(\omega)\sin\omega x\right)d\omega$, where $A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x\,dx$, $B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x\,dx$. At a point where $f(x)$ is discontinuous, the value of the Fourier integral equals the average of the left and right hand limits of $f(x)$ at that point.

Q. Find the Fourier integral representation of the fn

$$f(x) = \begin{cases} 1 & \text{if } |x| \le 1 \\ 0 & \text{if } |x| > 1 \end{cases}. \text{ Hence evaluate } \int_0^{\infty}\frac{\sin\lambda\cos x\lambda}{\lambda}\,d\lambda.$$

Ans. The Fourier integral representation of $f(x)$ is

$$f(x) = \int_0^{\infty}\left(A(\omega)\cos\omega x + B(\omega)\sin\omega x\right)d\omega, \text{ where}$$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x\,dx = \frac{1}{\pi}\int_{-1}^{1} 1\cdot\cos\omega x\,dx = \frac{1}{\pi}\left(\frac{\sin\omega x}{\omega}\right)_{-1}^{1}$$

$$= \frac{1}{\pi\omega}\left(\sin\omega - \sin(-\omega)\right) = \frac{2}{\pi\omega}\sin\omega.$$

and $B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x\,dx = \frac{1}{\pi}\int_{-1}^{1} 1\cdot\sin\omega x\,dx = -\frac{1}{\pi}\left.\frac{\cos\omega x}{\omega}\right)_{-1}^{1}$

$= -\frac{1}{\pi\omega}\left(\cos\omega - \cos(-\omega)\right) = \frac{-1}{\pi\omega}\left(\cos\omega - \cos\omega\right) = 0.$

$\therefore\ f(x) = \int_0^{\infty} \frac{2}{\pi\omega}\sin\omega\cos\omega x\,d\omega$, which is the Fourier integral representation of $f(x)$.

Here $x=1$ is a point of discontinuity of $f(x)$. The value of the integral is equal to $\frac{1}{2}\left[f(1^+) + f(1^-)\right]$, where

$f(1^+) = 0, \quad f(1^-) = 1.$

$\therefore$ value of the integral is $\frac{1}{2}(0+1) = 1/2.$

$$\therefore\ \frac{1}{2} = \frac{2}{\pi}\int_0^{\infty}\frac{\sin\omega\cos\omega}{\omega}\,d\omega.$$

$$\therefore\ \int_0^{\infty}\frac{\sin\omega\cos\omega}{\omega}\,d\omega = \pi/4.$$

Q. Find the FIR of the f $\quad f(x) = \begin{cases} 2 & \text{if } |x|\le 1 \\ 0 & \text{if } |x|>1 \end{cases}$

Ans The Fourier integral is

$$f(x) = \int_0^{\infty}\left(A(\omega)\cos\omega x + B(\omega)\sin\omega x\right)d\omega, \text{ where}$$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x\,dx = \frac{1}{\pi}\int_{-1}^{1} 2\cos\omega x\,dx = \frac{2}{\pi}\left.\frac{\sin\omega x}{\omega}\right)_{-1}^{1}$$

$$= \frac{4}{\pi\omega}\sin\omega$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x\,dx = \frac{1}{\pi}\int_{-1}^{1} 2\sin\omega x\,dx = \frac{2}{\pi}\left(-\frac{\cos\omega x}{\omega}\right)_{-1}^{1} = 0.$$

$$\therefore\ f(x) = \int_0^{\infty}\frac{4\sin\omega}{\pi\omega}\cos\omega x\,d\omega$$

$$\therefore\ f(x) = \frac{4}{\pi}\int_0^{\infty}\frac{\sin\omega\cos\omega x}{\omega}\,d\omega.$$

Fourier Cosine integral and Fourier Sine Integral

If f has a Fourier integral representation and is even, then $B(\omega)=0$. So the Fourier cosine integral is

$$f(x) = \int_0^{\infty} A(\omega)\cos\omega x\,d\omega, \text{ where } A(\omega) = \frac{2}{\pi}\int_0^{\infty} f(x)\cos\omega x\,dx$$

If f has a Fourier integral representation and is odd, then $A(\omega)=0$. So the Fourier Sine integral is

$$f(x) = \int_0^{\infty} B(\omega)\sin\omega x\,d\omega, \text{ where } B(\omega) = \frac{2}{\pi}\int_0^{\infty} f(x)\sin\omega x\,dx.$$

Q. Find the Fourier Sine and cosine integral of

$$f(x) = \begin{cases}\sin x; & 0\le x\le \pi \\ 0 & x>\pi\end{cases}$$

$\therefore$ Fourier Sine integral is $f(x) = \int_0^\infty B(\omega)\sin\omega x\, d\omega$, where

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\, dx$$

$$= \frac{2}{\pi}\int_0^\pi \sin x\sin\omega x\, dx = -\frac{1}{\pi}\int_0^\pi -2\sin\omega x\sin x\, dx$$

$$= -\frac{1}{\pi}\int_0^\pi \left(\cos(1+\omega)x - \cos(1-\omega)x\right) dx$$

$$= -\frac{1}{\pi}\left[\frac{\sin(1+\omega)x}{1+\omega} - \frac{\sin(1-\omega)x}{1-\omega}\right]_0^\pi = -\frac{1}{\pi}\left[\frac{\sin(1+\omega)\pi}{1+\omega} - \frac{\sin(1-\omega)\pi}{1-\omega}\right]$$

$$= -\frac{1}{\pi}\left[\frac{(1-\omega)\left(\cancel{\sin\pi\cos\omega\pi} + \cos\pi\sin\omega\pi\right) - (1+\omega)\left(\cancel{\sin\pi\cos\omega\pi} - \cos\pi\sin\omega\pi\right)}{1-\omega^2}\right]$$

$$= -\frac{1}{\pi}\left[\frac{(\omega-1)\sin\omega\pi - (1+\omega)\sin\omega\pi}{1-\omega^2}\right] = \frac{1}{\pi}\cdot\frac{2\sin\omega\pi}{1-\omega^2}$$

$$\therefore f(x) = \int_0^\infty \frac{2\sin\omega\pi}{\pi(1-\omega^2)}\sin\omega x\, d\omega$$

i.e. $f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin\omega\pi}{1-\omega^2}\sin\omega x\, d\omega$

Fourier Cosine integral is $f(x) = \int_0^\infty A(\omega)\cos\omega x\, d\omega$, where

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\, dx = \frac{2}{\pi}\int_0^\pi \sin x\cos\omega x\, dx$$

$$= \frac{1}{\pi}\int_0^\pi 2\cos\omega x\sin x\, dx = \frac{1}{\pi}\int_0^\pi \left(\sin(1+\omega)x + \sin(1-\omega)x\right) dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos(1+\omega)x}{1+\omega} \mp \frac{\cos(1-\omega)x}{1-\omega}\right]_0^\pi$$

$$= \frac{1}{\pi}\left[\left(-\frac{\cos(1+\omega)\pi}{1+\omega} \mp \frac{\cos(1-\omega)\pi}{1-\omega}\right) - \left(-\frac{1}{1+\omega} \mp \frac{1}{1-\omega}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{-(1-\omega)\left(\cos\pi\cos\omega\pi - \cancel{\sin\pi\sin\omega\pi}\right) \mp (1+\omega)\left(\cos\pi\cos\omega\pi + \cancel{\sin\pi\sin\omega\pi}\right)}{1-\omega^2} - \left(\frac{\cancel{1}+\omega \mp \cancel{1}\mp\omega}{1-\omega^2}\right)\right]$$

$$= \frac{1}{\pi(1-\omega^2)}\left[(1-\omega)\cos\omega\pi + (1+\omega)\cos\omega\pi + 2\right]$$

$$= \frac{1}{\cancel{\pi(1-\omega^2)}}\left(-\cancel{2\omega\cos\omega\pi - 2\omega}\right) = \frac{1}{\pi(1-\omega^2)}\left(2\cos\omega\pi + 2\right)$$

$$\therefore f(x) = \int_0^\infty \frac{2}{\pi(1-\omega^2)}\left(\cos\omega\pi + 1\right)\cos\omega x\, d\omega$$

i.e. $f(x) = \frac{2}{\pi}\int_0^\infty \frac{1+\cos\omega\pi}{1-\omega^2}\cos\omega x\, d\omega.$

Q. Using Fourier integral, Prove that $\int_0^\infty \frac{\omega^3 \sin x\omega}{\omega^4+4} d\omega = \frac{1}{2}\pi e^{-x}\cos x$, $x>0$

Ans. The Fourier integral of $f(x)$ is

$$f(x) = \int_0^\infty \left(A(\omega)\cos\omega x + B(\omega)\sin\omega x\right) d\omega \text{, where } A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x\, dx$$

and $B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x\, dx$.

Here we have to find the Fourier Sine integral of $e^{-x}\cos x$, $x>0$.

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\, dx = \frac{2}{\pi}\int_0^\infty e^{-x}\cos x \sin\omega x\, dx$$

$$= \frac{1}{\pi}\int_0^\infty e^{-x}\cdot 2\sin\omega x\cos x\, dx \qquad = \frac{1}{\pi}\int_0^\infty e^{-x}\left[\sin(1+\omega)x \mp \sin(1-\omega)x\right] dx$$

$$= \frac{1}{\pi}\left[\int_0^\infty e^{-x}\sin(1+\omega)x\, dx - \int_0^\infty e^{-x}\sin(1-\omega)x\, dx\right]$$

$$\boxed{\int_0^\infty e^{-ax}\sin bx\, dx = \frac{b}{a^2+b^2} \ \& \ \int_0^\infty e^{-ax}\cos bx\, dx = \frac{a}{a^2+b^2}}$$

$$\boxed{\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}}$$

$$= \frac{1}{\pi}\left[\frac{1+\omega}{1+(1+\omega)^2} - \frac{1-\omega}{1+(1-\omega)^2}\right] = \frac{1}{\pi}\left[\frac{1+\omega}{2+2\omega+\omega^2} - \frac{1-\omega}{2-2\omega+\omega^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{(1+\omega)(2+\omega^2-2\omega) - (1-\omega)(2+\omega^2+2\omega)}{(2+\omega^2+2\omega)(2+\omega^2-2\omega)}\right]$$

$$= \frac{1}{\pi}\left[\frac{\cancel{2}+\cancel{\omega^2}-\cancel{2\omega}+\cancel{2\omega}+\omega^3-\cancel{2\omega^2}-\cancel{2}-\cancel{\omega^2}-\cancel{2\omega}+\cancel{2\omega}+\omega^3+\cancel{2\omega^2}}{(2+\omega^2)^2-4\omega^2}\right]$$

$$\boxed{\begin{array}{l}\int_0^\infty \frac{\sin mx}{x}dx = \pi/2,\ m>0 \\ \int_0^\infty \frac{\sin^2 x}{x^2}dx = \pi/2\end{array}}$$

$$= \frac{1}{\pi}\left[\frac{2\omega^3}{4+\omega^4}\right]$$

$\therefore$ Fourier integral of $f(x) = e^{-x}\cos x$ is

$$e^{-x}\cos x = \int_0^\infty \frac{2\omega^3}{\pi(4+\omega^4)}\sin\omega x\, d\omega$$

$$\boxed{\begin{array}{l}\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) \\ \int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\end{array}}$$

$$\therefore \int_0^\infty \frac{\omega^3\sin\omega x}{\omega^4+4}d\omega = \frac{\pi}{2}e^{-x}\cos x.$$

Q. Using Fourier integral, Show that $\int_0^\infty \frac{1-\cos\pi\alpha}{\alpha}\sin x\alpha = \begin{cases}\pi/2; & 0<x<\pi \\ 0, & x>\pi\end{cases}$

Ans. Consider the fn $f(x) = \begin{cases}\pi/2, & 0<x<\pi \\ 0, & x>\pi\end{cases}$

Here we have to find the Fourier Sine integral of $f(x)$.

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\,dx = \frac{2}{\pi}\int_0^{\pi} \frac{\pi}{2}\sin\omega x\,dx = \frac{2}{\pi}\cdot\frac{\pi}{2}\left(-\frac{\cos\omega x}{\omega}\right)_0^{\pi}$$

$$= -\frac{1}{\omega}\left(\cos\omega\pi - 1\right) = \frac{1-\cos\omega\pi}{\omega}.$$

$$\therefore f(x) = \int_0^\infty \frac{1-\cos\omega\pi}{\omega}\sin\omega x\,d\omega$$

$$\text{i.e.,}\quad \int_0^\infty \frac{1-\cos\pi\alpha}{\alpha}\sin\alpha x\,d\alpha = \begin{cases} \pi/2, & 0<x<\pi \\ 0, & x>\pi \end{cases}$$

B. Find the Fourier integral of $f(x) = \begin{cases} 1-x^2; & |x|\le 1 \\ 0, & |x|>1 \end{cases}$ $\quad -1\le x\le 1$

Hence evaluate $\int_0^\infty \frac{\sin x - x\cos x}{x^3}\cos\frac{1}{2}x\,dx = 3\pi/16$

Ans. Here $f(x)$ is even.

$\therefore$ Fourier integral of $f(x)$ is $f(x) = \int_0^\infty A(\omega)\cos\omega x\,d\omega$, where

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\,dx = \frac{2}{\pi}\int_0^1 (1-x^2)\cos\omega x\,dx$$

$$= \frac{2}{\pi}\left[(1-x^2)\frac{\sin\omega x}{\omega} - (-2x)\left(-\frac{\cos\omega x}{\omega^2}\right) + (-2)\left(-\frac{\sin\omega x}{\omega^3}\right)\right]_0^1$$

$$= \frac{2}{\pi}\left[-2\frac{\cos\omega}{\omega^2} + 2\frac{\sin\omega}{\omega^3}\right] = \frac{4}{\pi}\left[\frac{\sin\omega - \omega\cos\omega}{\omega^3}\right]$$

$$\therefore f(x) = \int_0^\infty \frac{4}{\pi}\left(\frac{\sin\omega - \omega\cos\omega}{\omega^3}\right)\cos\omega x\,d\omega$$

At $x = 1/2$, $f(x)$ is continuous.

$$f(1/2) = \frac{4}{\pi}\int_0^\infty \frac{\sin\omega - \omega\cos\omega}{\omega^3}\cos\tfrac{1}{2}\omega\,d\omega$$

But $f(1/2) = 1 - 1/4 = 3/4$

$$\therefore \frac{3}{4} = \frac{4}{\pi}\int_0^\infty \frac{\sin\omega - \omega\cos\omega}{\omega^3}\cos\tfrac{1}{2}\omega\,d\omega$$

$$\text{i.e.,}\quad \int_0^\infty \frac{\sin\omega - \omega\cos\omega}{\omega^3}\cos\tfrac{1}{2}\omega\,d\omega = 3\pi/16$$

$$\text{i.e.,}\quad \int_0^\infty \frac{\sin x - x\cos x}{x^3}\cos\tfrac{1}{2}x\,dx = 3\pi/16$$

$\sin x\cos y = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$

$\cos x\cos y = \frac{1}{2}\left[\cos(x+y) + \cos(x-y)\right]$

$\sin x\sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$

Fourier Transform

1) The Fourier transform of the fn $f(x)$ is given by

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx = F(s), \text{ a fn of } s.$$

The inverse fourier transform is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(f(x))\, e^{-isx}\, ds$$

2) The fourier transform of an even fn $f(x)$ is, <u>Fourier Cosine transform</u> is given by

$$F_c[f(x)] \text{ or } F(f(x)) = \sqrt{2/\pi} \int_0^{\infty} f(x) \cos sx\, dx$$

Its inverse fourier transform is given by

$$f(x) = \sqrt{2/\pi} \int_0^{\infty} F[f(x)] \cos sx\, ds.$$

3) The fourier transform of an odd fn $f(x)$ is, <u>Fourier Sine transform</u> is given by

$$F_s(f(x)) \text{ or } F[f(x)] = \sqrt{2/\pi} \int_0^{\infty} f(x) \sin sx\, dx$$

Its inverse fourier transform is given by

$$f(x) = \sqrt{2/\pi} \int_0^{\infty} F[f(x)] \sin sx\, ds.$$

Pr. Find the fourier transform of $f(x) = 1$ in $|x|<1$; 0, otherwise. Hence evaluate $\int_0^{\infty} \frac{\sin x}{x} dx$

KTU **Ans.** Here $f(x) = 1, -1<x<1$ is an even fn.

$$\therefore F[f(x)] = \sqrt{2/\pi} \int_0^1 f(x) \cos sx\, dx = \sqrt{\tfrac{2}{\pi}} \int_0^1 1 \cdot \cos sx\, dx$$

$$= \sqrt{\tfrac{2}{\pi}} \cdot \left.\frac{\sin sx}{s}\right)_0^1 = \sqrt{\tfrac{2}{\pi}} \cdot \frac{\sin s}{s}.$$

Inverse fourier transform is

$$f(x) = \sqrt{\tfrac{2}{\pi}} \int_0^{\infty} F[f(x)] \cos sx\, ds = \sqrt{\tfrac{2}{\pi}} \int_0^{\infty} \sqrt{\tfrac{2}{\pi}} \frac{\sin s}{s} \cos sx\, ds. \text{ But } f(0) = 1. \quad \therefore \frac{2}{\pi}\int_0^{\infty} \frac{\sin s}{s} ds = 1, \quad \therefore \int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$$

OR

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} 1 \cdot e^{isx}\, dx = \frac{1}{\sqrt{2\pi}} \cdot \left.\frac{e^{isx}}{is}\right)_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}\, is}\left(e^{is} - e^{-is}\right) = \frac{1}{\sqrt{2\pi}\, is}\left(\cancel{\cos s} + i\sin s - \cancel{\cos s} + i\sin s\right)$$

$$= \frac{1}{\sqrt{2\pi}\, is} 2i\sin s = \sqrt{\tfrac{2}{\pi}}\, \frac{\sin s}{s}.$$

Pr. Find the fourier transform of $f(x) = e^{-ax}, x>0$; $= 0, x<0$; $a>0$.

Ans. $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-ax} e^{isx} dx = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-(a-is)x} dx$

$$= \frac{1}{\sqrt{2\pi}} \left. \frac{e^{-(a-is)x}}{-(a-is)} \right]_0^\infty = \frac{1}{\sqrt{2\pi}}\left[0 + \frac{1}{a-is}\right] = \frac{1}{\sqrt{2\pi}\,(a-is)}$$

Pr Find the Fourier transform of $f(x) = \begin{cases} 1-x^2 ; & |x|<1 \\ 0 ; & |x|>1 \end{cases}$. Hence evaluate $\int_0^\infty \frac{x\cos x - \sin x}{x^3} \cos\frac{x}{2}\, dx$.

An. Given $f(x) = \begin{cases} 1-x^2 ; & -1<x<1 \\ 0 ; & \text{otherwise} \end{cases}$. Here $f(x)$ is an even fn.

$$\therefore F[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^1 (1-x^2)\cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[(1-x^2)\frac{\sin sx}{s} - (-2x)\left(-\frac{\cos sx}{s^2}\right) + (-2)\left(-\frac{\sin sx}{s^3}\right) \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{2\cos s}{s^2} + \frac{2\sin s}{s^3} \right] = \sqrt{\frac{2}{\pi}} \cdot \frac{2}{s^3}\left[\sin s - s\cos s\right]$$

The inverse Fourier transform is given by

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F[f(x)] \cos sx\, ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}}\frac{2}{s^3}(\sin s - s\cos s)\cos sx\, ds$$

$$= \frac{4}{\pi} \int_0^\infty \frac{\sin s - s\cos s}{s^3} \cos sx\, ds .$$

At $x = 1/2$, $f(x)$ is continuous and $f(1/2) = 1 - 1/4 = 3/4$.

$$\therefore \frac{3}{4} = \frac{4}{\pi} \int_0^\infty \frac{\sin s - s\cos s}{s^3} \cos(s/2)\, ds .$$

$$\therefore \int_0^\infty \frac{\sin x - x\cos x}{x^3} \cos(x/2)\, dx = 3\pi/16 .$$

$$\therefore \int_0^\infty \frac{x\cos x - \sin x}{x^3} \cos(x/2)\, dx = -3\pi/16$$

Pr Find the Fourier Sine transform of e^{-x}; $x \geq 0$. Hence find $\int_0^\infty \frac{x\sin mx}{1+x^2}\, dx$.

(OR. Find the Fourier Sine transform of $e^{-|x|}$; $x \geq 0$)

An $F_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\sin sx\, dx = \sqrt{2/\pi}\int_0^\infty e^{-x}\sin sx\, dx = \sqrt{\frac{2}{\pi}}\frac{s}{s^2+1}$.

By the inverse Fourier Sine transform,

$$f(x) = \sqrt{2/\pi} \int_0^\infty F_s(f(x))\sin sx\, ds = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{s}{s^2+1}\sin sx\, ds$$

$$= \frac{2}{\pi}\int_0^\infty \frac{s}{s^2+1}\sin sx\, ds$$

$\therefore \; e^{-x} = \frac{2}{\pi}\int_0^\infty \frac{s}{s^2+1} \sin sx \, ds$

$\therefore \; e^{-m} = \frac{2}{\pi}\int_0^\infty \frac{s}{s^2+1} \sin ms \, ds$ $\therefore \int_0^\infty \frac{s}{s^2+1} \sin ms \, ds = \frac{\pi}{2} e^{-m}$

$\therefore \int_0^\infty \frac{x}{x^2+1} \sin mx \, dx = \frac{\pi}{2} e^{-m}$

Q. Find the Fourier sine and cosine transform of e^{-ax}; $a>0, x>0$

Ans. $F_c(f(x)) = \sqrt{2/\pi}\int_0^\infty f(x) \cos sx \, dx = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax} \cos sx \, dx$

$= \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+s^2}$ $\left(\because \int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2+b^2}\right)$

$F_s(f(x)) = \sqrt{2/\pi}\int_0^\infty f(x) \sin sx \, dx = \sqrt{2/\pi}\int_0^\infty e^{-ax} \sin sx \, dx$

$= \sqrt{\frac{2}{\pi}} \; \frac{s}{s^2+a^2}$ $\left(\because \int_0^\infty e^{-ax} \sin bx \, dx = \frac{b}{a^2+b^2}\right)$

Q. Find the Fourier transform of $f(x) = \begin{cases} x & \text{for } |x| \le a \\ 0 & \text{for } |x| > a \end{cases}$

Ans. $f(x)$ is an odd fn.

$\therefore \; F[f(x)] = \sqrt{2/\pi}\int_0^a f(x) \sin sx \, dx = \sqrt{2/\pi}\int_0^a x \cdot \sin sx \, dx$

$= \sqrt{2/\pi}\left[x\left(-\frac{\cos sx}{s}\right) - (1)\left(-\frac{\sin sx}{s^2}\right)\right]_0^a$

$= \sqrt{\frac{2}{\pi}}\left[-a\frac{\cos as}{s} + \frac{\sin as}{s^2}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{\sin as - as\cos as}{s^2}\right]$

Properties of Fourier Transform

1. Linearity of Fourier transform

i, $F[a f(x) + b g(x)] = a F[f(x)] + b F[g(x)]$

Pf. $F[a f(x) + b g(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty [a f(x) + b g(x)] e^{isx} \, dx$

$= a \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(x) e^{isx} dx + b \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty g(x) e^{isx} dx$

$= a F[f(x)] + b F[g(x)]$

2. Shifting property

If $F[f(x)] = F(s)$, then $F[f(x-a)] = e^{isa} F(s)$

Pf. $F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a)\, e^{isx}\, dx$. put $x-a=t$, $dx = dt$.

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{is(a+t)}\, dt$

$= e^{isa} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt = e^{isa}\, F[f(t)]$

$= e^{isa}\, F(s)$

3) change of scale property.

If $F[f(x)] = F(s)$, then $F[f(ax)] = \frac{1}{|a|} F(s/a)$; $a \neq 0$.

Pf $F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax)\, e^{isx}\, dx$ put $\frac{ax = t}{a\,dx = dt}$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{is(t/a)} \cdot \frac{dt}{a} = \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i(\frac{s}{a})t}\, dt$

$= \frac{1}{a} F(s/a)$; $a > 0$.

Also $F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(t)\, e^{i(\frac{s}{a})t} \frac{dt}{a}$; if $a < 0$.

$= -\frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i(\frac{s}{a})t}\, dt = -\frac{1}{a} \cdot F(s/a)$; $a<0$

$\therefore F[f(ax)] = \frac{1}{|a|} F(s/a)$; $a \neq 0$

4) If $F[f(x)] = F(s)$, then $F[x^n f(x)] = (-i)^n \frac{d^n}{ds^n} F(s)$.

Pf We have $F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

diff w.r.to s,

$\frac{d}{ds} F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} (ix)\, dx$

differentiate again w.r.to s,

$\frac{d^2}{ds^2} F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} (ix)^2\, dx$

Proceeding like these,

$\frac{d^n}{ds^n} F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} (ix)^n\, dx$

$= (i)^n \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n f(x)\, e^{isx}\, dx = (i)^n F[x^n f(x)]$

$\therefore F[x^n f(x)] = \dfrac{1}{(i)^n}\dfrac{d^n}{ds^n}F(s) = (-i)^n \dfrac{d^n}{ds^n}F(s).$

5) If $F[f(x)] = F(s)$, then $F[e^{iax} f(x)] = F(s+a)$, &

$F[e^{-iax} f(x)] = F(s-a)$.

Pf $F[e^{iax} f(x)] = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} e^{iax} f(x) e^{isx}\,dx = \dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{i(s+a)x}\,dx$

$= F(s+a).$

$F[e^{-iax} f(x)] = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} e^{-iax} f(x) e^{isx}\,dx = \dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{i(s-a)x}\,dx$

$= F(s-a).$

6) Modulation Property

If $F[f(x)] = F(s)$ then $F[f(x)\cos ax] = \frac{1}{2}[F(s-a) + F(s+a)]$

& $F[f(x)\sin ax] = \frac{1}{2i}[F(s+a) - F(s-a)]$.

Pf $F[f(x)\cos ax] = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} f(x)\cos ax\, e^{isx}\,dx$

$= \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} f(x)\left[\dfrac{e^{iax}+e^{-iax}}{2}\right] e^{isx}\,dx$

$= \dfrac{1}{2}\left[\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} f(x) e^{i(s+a)x}\,dx + \dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{i(s-a)x}\,dx\right]$

$= \frac{1}{2}\left[F(s+a) + F(s-a)\right]$

Also $F[f(x)\sin ax] = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} f(x)\sin ax\, e^{isx}\,dx$

$= \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} f(x)\left(\dfrac{e^{iax}-e^{-iax}}{2i}\right) e^{isx}\,dx$

$= \dfrac{1}{2i}\left[\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} f(x) e^{i(s+a)x}\,dx - \dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{i(s-a)x}\,dx\right]$

$= \dfrac{1}{2i}\left[F(s+a) - F(s-a)\right]$

7) Parseval's Identity : $\displaystyle\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |F(s)|^2\,ds.$

— o —

Q: Find the Fourier transform of e^{-x^2}. Then find $F[x e^{-x^2}]$.

Ans:

$$F[e^{-x^2}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2} \cdot e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(x^2 - isx + \left(\frac{is}{2}\right)^2 - \left(\frac{is}{2}\right)^2\right)}\, dx$$

$$= e^{-s^2/4} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(x - \frac{is}{2}\right)^2} dx \qquad \text{put } x - \frac{is}{2} = t, \quad dx = dt$$

$$= e^{-s^2/4} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2}\, dt$$

$$= e^{-s^2/4} \cdot \frac{1}{\sqrt{2\pi}} \cdot 2 \int_0^{\infty} e^{-t^2}\, dt$$

$$= e^{-s^2/4} \cdot \frac{1}{\sqrt{2\pi}} \cdot 2 \cdot \frac{\sqrt{\pi}}{2} \qquad \left(\because \int_0^{\infty} e^{-x^2}\, dx = \sqrt{\pi}/2\right)$$

$$= \frac{1}{\sqrt{2}} e^{-s^2/4}$$

We have $F[x^n f(x)] = (-i)^n \frac{d^n}{ds^n} F(s)$, where $F(s) = F[f(x)] = \frac{1}{\sqrt{2}} e^{-s^2/4}$.

$$\therefore F[x f(x)] = (-i) \frac{d}{ds} F(s),$$

$$\therefore F[x \cdot e^{-x^2}] = (-i) \frac{d}{ds}\left[\frac{1}{\sqrt{2}} e^{-s^2/4}\right] = (-i) \frac{1}{\sqrt{2}} e^{-s^2/4} \cdot \left(\frac{-2s}{4}\right)$$

$$= \frac{+is}{2\sqrt{2}} e^{-s^2/4}$$

* A fn $f(x)$ is ~~said to be self~~ reciprocal of the Fourier transform of $f(x)$ is $f(s)$.

i.e., $F[f(x)] = f(s)$.

Q: Show that the fn $f(x) = e^{-x^2/2}$ is self reciprocal.

Ans:

$$F[e^{-x^2/2}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{isx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 - 2isx)}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 - 2isx + (is)^2 - (is)^2)}\, dx = e^{-s^2/2} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x - is)^2}{2}}\, dx$$

$$= e^{-s^2/2} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} \sqrt{2}\, dt$$

put $\frac{x-is}{\sqrt{2}} = t$

$dx = \sqrt{2}\, dt$

$$= e^{-s^2/2} \cdot \frac{1}{\sqrt{\pi}} \cdot 2 \int_{0}^{\infty} e^{-t^2} dt = e^{-s^2/2} \cdot \frac{1}{\sqrt{\pi}} \cdot 2 \cdot \frac{\sqrt{\pi}}{2}$$

$$= e^{-s^2/2} = f(s).$$

$\therefore$ $F[f(x)] = f(s)$ $\therefore$ $f(x)$ is Self reciprocal.

Now $\cancel{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \cdot e^{isx} dx = e^{-s^2/2}}$.

i.e. $\cancel{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} (\cos sx + i \sin sx) dx = e^{-s^2/2}}$.

i.e. $\cancel{\sqrt{2/\pi} \int_{0}^{\infty} e^{-x^2/2} \cos sx\, dx + \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \sin sx\, dx = e^{-s^2/2}}$,

~~equating real and imaginary parts,~~

$\cancel{\sqrt{2/\pi} \int_{0}^{\infty} e^{-x^2/2} \cos sx\, dx = e^{-s^2/2}}$ & $\cancel{\frac{\sqrt{2/\pi}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \sin sx\, dx = 0}$

i.e. $\cancel{F_c[e^{-x^2/2}] = e^{-s^2/2}}$ & $\cancel{F_s(e^{-x^2/2}) = 0}$.

Q Find the Fourier Sine transform of $f(x) = 1/x$.

Ans. $F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx\, dx = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{1}{x} \cdot \sin sx\, dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\sin sx}{x} dx = \sqrt{\frac{2}{\pi}} \cdot \pi/2 \quad \left(\because \int_0^{\infty} \frac{\sin mx\, dx}{x} = \pi/2\right)$$

$$= \underline{\sqrt{\pi/2}}$$

Q Find the Fourier Cosine transform of $f(x) = \frac{1}{1+x^2}$.

Ans $F_c\left[\frac{1}{1+x^2}\right] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{1}{1+x^2} \cos sx\, dx = I$ (say) —— (1)

$$\frac{dI}{ds} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{1}{1+x^2} \cdot -\sin sx \cdot x\, dx$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{x^2}{x(1+x^2)} \sin sx\, dx$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{(1+x^2-1)}{x(1+x^2)} \sin sx\, dx$$

$$I = -\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin sx}{x}\,dx + \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin sx}{x(1+x^2)}\,dx$$

$$\frac{dI}{ds} = -\sqrt{\frac{2}{\pi}} \cdot \pi/2 + \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin sx}{x(1+x^2)}\,dx \quad \text{---} (2) \quad \left(\because \int_0^\infty \frac{\sin mx}{x}\,dx = \pi/2\right)$$

$$\frac{d^2I}{ds^2} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{x(1+x^2)} \cos sx \cdot \cancel{x}\,dx = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\cos sx}{1+x^2}\,dx = I$$

$\therefore (D^2-1)I = 0$; $D \equiv d/ds$

Auxiliary eqn is $m^2-1=0$. $m = \pm 1$.

Soln is $\underline{I = Ae^{s} + Be^{-s}}$. $\quad \frac{dI}{ds} = Ae^{s} - Be^{-s}$

when $s=0$, $I = A+B$

Also when $s=0$, $I = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{1+x^2}\,dx = \sqrt{\frac{2}{\pi}} \left(\tan^{-1}x\right)_0^\infty = \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2} = \sqrt{\pi/2}$.

$\therefore \underline{A+B = \sqrt{\pi/2}}$

when $s=0$, $\frac{dI}{ds} = A-B$.

Also when $s=0$, $\frac{dI}{ds} = -\sqrt{\frac{2}{\pi}} \cdot \pi/2 = -\sqrt{\pi/2}$

$\therefore \underline{A-B = -\sqrt{\pi/2}}$

Solving, $2A = 0$ $\quad \therefore A = 0$.

$B = \sqrt{\pi/2}$.

$\therefore I = \sqrt{\pi/2}\, e^{-s}$.

$\therefore \underline{F_c\left[\frac{1}{1+x^2}\right] = \sqrt{\pi/2}\, e^{-s}}$

Pr Find the Cosine transform of $f(x) = 1$ if $0<x<1$, $f(x) = -1$ if $1<x<2$, $f(x) = 0$ if $x>2$.

An. $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\,dx$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 1.\cos sx\,dx + \int_1^2 -1 \cos sx\,dx\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\left(\frac{\sin sx}{s}\right)_0^1 - \left(\frac{\sin sx}{s}\right)_1^2\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin s}{s} - \frac{\sin s2}{s} + \frac{\sin s}{s}\right] = \sqrt{\frac{2}{\pi}} \left[\frac{2\sin s}{s} - \frac{\sin 2s}{s}\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2\sin s - \sin 2s}{s}\right]$$

Q. Does the Fourier Cosine transform of e^x; $0<x<\infty$ exist.

An. $F_c[e^x] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^x \cos sx \, dx$

$= \sqrt{\frac{2}{\pi}} \left[\frac{e^x}{1+s^2} \left(\cos sx + s \sin sx \right) \right]_0^\infty$

But $\lim_{x \to \infty} e^x (\cos sx + s \sin sx)$ does not exist. $\left(\because \int_0^\infty e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx) \right)$

$\therefore F_c[e^x]$ does not exist.

Q. Find the fourier Cosine transform of $f(x) = x$ in $0<x<2$, $f(x) = 0$ in $x>2$.

An. $F_c[f(x)] = \sqrt{2/\pi} \int_0^\infty f(x) \cos sx \, dx = \sqrt{2/\pi} \int_0^2 x \cdot \cos sx \, dx$

$= \sqrt{2/\pi} \left[x \cdot \frac{\sin sx}{s} - 1 \cdot \left(-\frac{\cos sx}{s^2} \right) \right]_0^2 = \sqrt{\frac{2}{\pi}} \left[\frac{2 \sin 2s}{s} + \frac{\cos 2s}{s^2} - \frac{1}{s^2} \right]$

Q. Find the Fourier Cosine transform of e^{-x^2}.

An. $F_c[e^{-x^2}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2} \cos sx \, dx$ —— (1)

Diff. w.r.t. s

$\frac{d}{ds} F_c[e^{-x^2}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2} (-\sin sx) \cdot x \, dx$

$= \frac{1}{2} \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2} (-2x) \sin sx \, dx$

$= \frac{1}{2} \sqrt{\frac{2}{\pi}} \int_0^\infty \sin sx \cdot \left(-e^{-x^2} \cdot 2x \right) dx$

$= \frac{1}{2} \sqrt{\frac{2}{\pi}} \left[\left(\sin sx \cdot e^{-x^2} \right)_0^\infty - \int_0^\infty s \cos sx \cdot e^{-x^2} dx \right]$

$= -\frac{s}{2} \sqrt{2/\pi} \int_0^\infty e^{-x^2} \cos sx \, dx = -\frac{s}{2} F_c(e^{-x^2})$

$\therefore \frac{d F_c(e^{-x^2})}{F_c(e^{-x^2})} = -\frac{1}{2} s \, ds.$

Integrating, $\log F_c(e^{-x^2}) = -s^2/4 + k.$

$\therefore F_c(e^{-x^2}) = k' e^{-s^2/4}.$ —— (2)

when $s=0$, (1) $\Rightarrow$ $F_c(e^{-x^2}) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2} dx = \sqrt{\frac{2}{\pi}} \cdot \frac{\sqrt{\pi}}{2} = \frac{1}{\sqrt{2}}$

(2) $\Rightarrow$ $F_c(e^{-x^2}) = k'$ $\quad \therefore k' = 1/\sqrt{2}$

$\therefore F_c(e^{-x^2}) = \frac{1}{\sqrt{2}} e^{-s^2/4}$

-8-

Pr: Solve the integral equation $\int_0^\infty f(x)\cos sx\,dx = \begin{cases} 1-s, & 0 \le s \le 1 \\ 0, & s > 1 \end{cases}$.

Hence deduce that $\int_0^\infty \frac{\sin^2 t}{t^2}\,dt = \pi/2$.

Ans: We have $F_c[f(x)] = \sqrt{2/\pi}\int_0^\infty f(x)\cos sx\,dx = \sqrt{\frac{2}{\pi}}(1-s);\ 0 \le s \le 1$.

By Inverse Fourier Cosine transform,

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c[f(x)]\cos sx\,ds = \sqrt{\frac{2}{\pi}}\int_{s=0}^{1}\sqrt{\frac{2}{\pi}}(1-s)\cos sx\,ds$$

$$= \frac{2}{\pi}\left[(1-s)\frac{\sin sx}{x} - (-1)\left(-\frac{\cos sx}{x^2}\right)\right]_{s=0}^{1}$$

$$= \frac{2}{\pi}\left[-\frac{\cos x}{x^2} + \frac{1}{x^2}\right] = \frac{2(1-\cos x)}{\pi x^2}$$

$$\therefore f(x) = \frac{2(1-\cos x)}{\pi x^2}$$

We have $\int_0^\infty f(x)\cos sx\,dx = 1-s;\ 0 \le s \le 1$.

$$\text{i.e., } \int_0^\infty \frac{2(1-\cos x)}{\pi x^2}\cos sx\,dx = 1-s.$$

when $s=0$, $\int_0^\infty \frac{2}{\pi x^2}(1-\cos x)\,dx = 1$.

$$\text{i.e., } \int_0^\infty \frac{1-\cos x}{x^2}\,dx = \pi/2$$

$$\text{i.e., } \int_0^\infty 2\frac{\sin^2 x/2}{x^2}\,dx = \pi/2.$$

put $x/2 = t$.
$dx = 2dt$

$$\text{i.e., } \int_0^\infty 2\cdot\frac{\sin^2 t}{4t^2}\cdot 2dt = \pi/2 \Rightarrow \int_0^\infty \frac{\sin^2 t}{t^2}\,dt = \pi/2.$$

Pr ST the fourier transform of $f(x) = \begin{cases} a-|x|; & |x|<a \\ 0, & |x|>a \end{cases}$ is

$\sqrt{\frac{2}{\pi}}\,\frac{1-\cos as}{s^2}$. Hence evaluate $\int_0^\infty \frac{\sin^2 t}{t^2}\,dt$ and $\int_0^\infty \frac{\sin^4 t}{t^4}\,dt$.

Ans: $f(x) = \begin{cases} a+x; & -a<x<0 \\ a-x; & 0<x<a \end{cases}$

$$\therefore F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-a}^{0}(a+x)e^{isx}\,dx + \frac{1}{\sqrt{2\pi}}\int_0^a (a-x)e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[(a+x)\frac{e^{isx}}{is} - 1\cdot\frac{e^{isx}}{(is)^2}\right]_{-a}^{0} + \frac{1}{\sqrt{2\pi}}\left[(a-x)\frac{e^{isx}}{is} - (-1)\frac{e^{isx}}{(is)^2}\right]_0^a$$

$$= \frac{1}{\sqrt{2\pi}}\left[\cancel{\frac{a}{is}} + \frac{1}{s^2} - \frac{e^{-ias}}{s^2} + -\frac{e^{ias}}{s^2} - \cancel{\frac{a}{is}} + \frac{1}{s^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2}{s^2} - \frac{1}{s^2}\cdot 2\cos as\right] = \sqrt{\frac{2}{\pi}}\,\frac{1-\cos as}{s^2}$$

The inverse Fourier transform is

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}}\ \frac{1-\cos as}{s^2}\ e^{-isx}\, ds$$

At $x=0$, $f(x)=a$

$$\therefore a = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \cdot \frac{1-\cos as}{s^2}\, ds = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{2\sin^2 as/2}{s^2}\, ds$$

$$= \frac{4}{\pi}\int_0^{\infty} \frac{\sin^2 as/2}{s^2}\, ds \qquad \text{put } \frac{as}{2} = t, \quad a\,ds = 2dt$$

$$= \frac{4}{\pi}\int_0^{\infty} \frac{\sin^2 t}{4t^2/a^2} \cdot \frac{2dt}{a}$$

$$a = \frac{a}{\pi}\cdot 2\int_0^{\infty} \frac{\sin^2 t}{t^2}\, dt \qquad \therefore \int_0^{\infty} \frac{\sin^2 t}{t^2}\, dt = \underline{\pi/2}$$

By Parseval's Identity,

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |F(s)|^2\, ds.$$

$$\text{i.e., } \int_{-a}^{a} (a-|x|)^2\, dx = \int_{-\infty}^{\infty} \frac{2}{\pi}\left(\frac{1-\cos as}{s^2}\right)^2 ds$$

$$\text{i.e., } \int_{-a}^{0} (a+x)^2\, dx + \int_0^{a} (a-x)^2\, dx = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{2\sin^2 as/2}{s^2}\right)^2 ds.$$

$$\text{i.e., } \left.\frac{(a+x)^3}{3}\right)_{-a}^{0} + \left.\frac{(a-x)^3}{-3}\right)_0^{a} = \frac{16}{\pi}\int_0^{\infty} \frac{\sin^4 as/2}{s^4}\, ds.$$

$$a^3/3 + a^3/3 = \frac{16}{\pi}\int_0^{\infty} \frac{\sin^4 t}{(2t/a)^4}\,\frac{2dt}{a}$$

$$\text{i.e., } \frac{2a^3}{3} = \frac{16}{\pi}\cdot a^3 \cdot \frac{2}{16}\int_0^{\infty} \frac{\sin^4 t}{t^4}\, dt \qquad \therefore \int_0^{\infty}\frac{\sin^4 t}{t^4}\, dt = \underline{\underline{\pi/3}}$$

Q/ Use Fourier integral to show that $\displaystyle\int_0^{\infty} \frac{\cos x\omega + \omega\sin x\omega}{1+\omega^2}\, d\omega$

KTU

$$= \begin{cases} 0 & \text{if } x<0 \\ \pi/2 & \text{if } x=0 \\ \pi e^{-x} & \text{if } x>0 \end{cases}$$

Ans. Consider the fn $f(x) = \begin{cases} \pi e^{-x} & ; x>0 \\ \pi/2 & ; x=0 \\ 0 & ; x<0 \end{cases}$

The Fourier integral of $f(x)$ is

$$f(x) = \int_0^{\infty} \big(A(\omega)\cos\omega x + B(\omega)\sin\omega x\big)\, d\omega, \quad \text{where}$$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x\, dx; \quad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x\, dx,$$

Now $A(\omega) = \frac{1}{\pi}\int_0^\infty \pi e^{-x} \cos\omega x \, dx = \int_0^\infty e^{-x} \cos\omega x \, dx = \frac{1}{1+\omega^2}$

$B(\omega) = \frac{1}{\pi}\int_0^\infty \pi e^{-x} \sin\omega x \, dx = \int_0^\infty e^{-x} \sin\omega x \, dx = \frac{\omega}{1+\omega^2}$

Since $\int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2+b^2}$; $\int_0^\infty e^{-ax} \sin bx \, dx = \frac{b}{a^2+b^2}$

$\therefore f(x) = \int_0^\infty \left(\frac{1}{1+\omega^2}\cos\omega x + \frac{\omega}{1+\omega^2}\sin\omega x\right) d\omega$

$= \int_0^\infty \frac{\cos\omega x + \omega \sin\omega x}{1+\omega^2} \, d\omega$

When $x=0$, $f(x) = \int_0^\infty \frac{1}{1+\omega^2} d\omega = \tan^{-1}\omega \Big)_0^\infty = \pi/2$.

$\therefore \int_0^\infty \frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2} d\omega = \begin{cases} 0 & \text{if } x<0 \\ \pi/2 & \text{if } x=0 \\ \pi e^{-x} & \text{if } x>0 \end{cases}$

Q. Represent $f(x) = \begin{cases} x^2 ; & 0<x<1 \\ 0 ; & x>1 \end{cases}$ as a Fourier Cosine integral.

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An. The Fourier Cosine integral of $f(x)$ is

$f(x) = \int_0^\infty A(\omega)\cos\omega x \, d\omega$; where $A(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x \, dx$.

ie, $A(\omega) = \frac{2}{\pi}\int_0^1 x^2 \cos\omega x \, dx$

$= \frac{2}{\pi}\left[x^2 \frac{\sin\omega x}{\omega} - (2x)\left(-\frac{\cos\omega x}{\omega^2}\right) + (2)\left(-\frac{\sin\omega x}{\omega^3}\right)\right]_0^1$

$= \frac{2}{\pi}\left[x^2\frac{\sin\omega x}{\omega} + 2x\frac{\cos\omega x}{\omega^2} - 2\frac{\sin\omega x}{\omega^3}\right]_0^1$

$= \frac{2}{\pi}\left[\frac{\sin\omega}{\omega} + 2\frac{\cos\omega}{\omega^2} - 2\frac{\sin\omega}{\omega^3}\right]$

~~$B(\omega)$~~ $= \frac{2}{\pi}$

$\therefore$ Fourier Cosine integral of $f(x)$ is

$f(x) = \int_0^\infty \frac{2}{\pi}\left[\frac{\sin\omega}{\omega} + \frac{2\cos\omega}{\omega^2} - \frac{2\sin\omega}{\omega^3}\right]\cos\omega x \, d\omega$

$= \frac{2}{\pi}\int_0^\infty \left[\left(1-\frac{2}{\omega^2}\right)\frac{\sin\omega}{\omega} + \frac{2\cos\omega}{\omega^2}\right]\cos\omega x \, d\omega$

$= \frac{2}{\pi}\int_0^\infty \left[\left(1-\frac{2}{\omega^2}\right)\sin\omega + \frac{2\cos\omega}{\omega}\right]\frac{\cos\omega x}{\omega} \, d\omega$.

Q. Express $f(x) = \begin{cases} 1 ; & 0 < x < \pi \\ 0, & x > \pi \end{cases}$ as a Fourier sine integral and

KTU evaluate $\int_0^\infty \frac{1 - \cos \pi \omega}{\omega} \cancel{\sin \pi\omega}\ \sin\omega \, d\omega$.

Ans. The Fourier sine integral of $f(x)$ is

$$f(x) = \int_0^\infty B(\omega) \sin \omega x \, d\omega, \text{ where } B(\omega) = \frac{2}{\pi} \int_0^\infty f(x) \sin \omega x \, dx$$

$$B(\omega) = \frac{2}{\pi} \int_0^\pi 1 \cdot \sin \omega x \, dx = -\frac{2}{\pi} \left. \frac{\cos \omega x}{\omega} \right)_0^\pi$$

$$= -\frac{2}{\pi} \left(\frac{\cos \pi\omega - 1}{\omega} \right) = \frac{2}{\pi} \left(\frac{1 - \cos \pi \omega}{\omega} \right)$$

$$\therefore f(x) = \int_0^\infty \frac{2}{\pi} \frac{(1 - \cos \pi\omega)}{\omega} \sin \omega x \, d\omega$$

At $x = 1$, $f(x)$ is continuous and $f(1) = 1$.

$$\therefore 1 = \frac{2}{\pi} \int_0^\infty \frac{1 - \cos \pi\omega}{\omega} \sin \omega \, d\omega$$

$$\therefore \int_0^\infty \frac{1 - \cos \pi \omega}{\omega} \sin \omega \, d\omega = \pi/2$$

Q. Using Fourier integral representation, show that

$$\int_0^\infty \frac{\sin \omega - \omega \cos \omega}{\omega^2} \sin \omega x \, d\omega = \begin{cases} \pi/2 \, x, & \text{if } 0 < x < 1 \\ \pi/4, & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases}$$

Ans. Consider the fn

KTU

$$f(x) = \begin{cases} \pi/2 \, x ; & 0 < x < 1 \\ \pi/4 ; & x = 1 \\ 0 ; & x > 1. \end{cases}$$

The Fourier sine integral of $f(x)$ is

$$f(x) = \int_0^\infty B(\omega) \sin \omega x \, d\omega, \text{ where } B(\omega) = \frac{2}{\pi} \int_0^\infty f(x) \sin \omega x \, dx$$

$$B(\omega) = \frac{2}{\pi} \int_0^1 \frac{\pi}{2} x \sin \omega x \, dx = \int_0^1 x \sin \omega x \, dx$$

$$= x \cdot \left(-\frac{\cos \omega x}{\omega} \right) - (1) \left(-\frac{\sin \omega x}{\omega^2} \right) \Big]_0^1$$

$$= -\frac{x \cos \omega x}{\omega} + \frac{\sin \omega x}{\omega^2} \Big]_0^1 = -\frac{\cos \omega}{\omega} + \frac{\sin \omega}{\omega^2} = \frac{\sin \omega - \omega \cos \omega}{\omega^2}$$

$$\therefore f(x) = \int_0^\infty \left(\frac{\sin \omega - \omega \cos \omega}{\omega^2} \right) \sin \omega x \, d\omega$$

ω.

$$\int_0^\infty \left(\frac{\sin\omega - \omega\cos\omega}{\omega^2}\right)\sin\omega x \, d\omega = \begin{cases} \frac{\pi}{2}x ; & 0<x<1 \\ \frac{\pi}{4} ; & x=1 \\ 0 ; & x>1 \end{cases}$$

Note: when $x=1$, Integral becomes

$$\int_0^\infty \left(\frac{\sin\omega - \omega\cos\omega}{\omega^2}\right)\sin\omega \, d\omega = \int_0^\infty \frac{\sin^2\omega}{\omega^2}\, d\omega - \int_0^\infty \frac{1}{2}\cdot\frac{\sin 2\omega}{\omega}\, d\omega$$

$$= \frac{\pi}{2} - \frac{1}{2}\cdot\frac{\pi}{2} \quad \left(\because \int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \pi/2 \text{ \& } \int_0^\infty \frac{\sin mx}{x}\,dx = \pi/2 ;\ m>0\right)$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \pi/4$$

Pr. Find the Fourier transform of $f(x) = \begin{cases} 1 & \text{if } |x|<1 \\ 0, & \text{otherwise} \end{cases}$

Ans. Given $f(x) = \begin{cases} 1 ; & -1<x<1 \\ 0, & \text{otherwise} \end{cases}$

Fourier transform of $f(x)$ is

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(x)\, e^{isx}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-1}^1 1.\ e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{e^{isx}}{is}\right)_{-1}^1 = \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{is}\left(e^{is} - e^{-is}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{1}{is}\cdot 2i\sin s = \sqrt{\frac{2}{\pi}}\,\frac{\sin s}{s}$$

OR

Here, $f(x) = \begin{cases} 1 ; & -1<x<1 \\ 0, & \text{otherwise} \end{cases}$ is an even fn.

$$\therefore F[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx = \sqrt{\frac{2}{\pi}}\int_0^1 1.\ \cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\cdot\left(\frac{\sin sx}{s}\right)_0^1 = \sqrt{\frac{2}{\pi}}\cdot\frac{\sin s}{s}.$$

Pr. Find the Fourier Cosine transform of

KTU

$$f(x) = \begin{cases} x^2 ; & \text{if } 0<x<1 \\ 0, & \text{if } x>1. \end{cases}$$

Ans. We have $F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx = \sqrt{\frac{2}{\pi}}\int_0^1 x^2\cos sx\,dx$

$$= \sqrt{\frac{2}{\pi}}\left[x^2\left(\frac{\sin sx}{s}\right) - (2x)\left(-\frac{\cos sx}{s^2}\right) + (2)\left(-\frac{\sin sx}{s^3}\right)\right]_0^1$$

$$= \sqrt{\frac{2}{\pi}}\left[x^2\frac{\sin sx}{s} + 2x\frac{\cos sx}{s^2} - 2\frac{\sin sx}{s^3}\right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin s}{s} + \frac{2 \cos s}{s^2} - \frac{2 \sin s}{s^3} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s^2 \sin s + 2s \cos s - 2 \sin s}{s^3} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{(s^2 - 2) \sin s + 2s \cos s}{s^3} \right]$$

1) Find the fourier sine transform of $e^{-|x|}$. Hence evaluate $\int_0^\infty \frac{\omega \sin \omega x}{1+\omega^2} d\omega$

An. $f(x) = e^{-|x|} \Rightarrow f(x) = e^{-x}; 0 < x < \infty$.

$$F_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cdot \sin sx \, dx = \sqrt{\frac{2}{\pi}} \cdot \frac{s}{s^2+1}.$$

By the inverse fourier sine transform,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(f(x)) \sin sx \, ds = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{s}{s^2+1} \sin sx \, ds$$

$$= \frac{2}{\pi} \int_0^\infty \frac{s}{s^2+1} \sin sx \, ds.$$

$$\therefore e^{-x} = \frac{2}{\pi} \int_0^\infty \frac{s}{s^2+1} \sin sx \, ds$$

$$\therefore \frac{\pi}{2} e^{-x} = \int_0^\infty \frac{s}{s^2+1} \sin xs \, ds \qquad \text{i.e., } \int_0^\infty \frac{s}{s^2+1} \sin xs \, ds = \frac{\pi}{2} e^{-x}$$

$$\therefore \int_0^\infty \frac{\omega}{\omega^2+1} \sin x\omega \, d\omega = \pi/2 \, e^{-x}$$

2) Find the fourier cosine transform of $f(x) = \sin x; \; 0 < x < \pi$.

An. $F_c(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^\pi \sin x \cos sx \, dx = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \int_0^\pi 2 \sin x \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \int_0^\pi \left[\sin(s+1)x - \sin(s-1)x\right] dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \left[-\frac{\cos(s+1)x}{s+1} + \frac{\cos(s-1)x}{s-1}\right]_0^\pi$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \left[\left(-\frac{\cos(s+1)\pi}{s+1} + \frac{\cos(s-1)\pi}{s-1}\right) - \left(\frac{-1}{s+1} + \frac{1}{s-1}\right)\right]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \left[\frac{-(s-1)\cos(s\pi+\pi) + (s+1)\cos(s\pi-\pi)}{s^2-1} - \left(\frac{\cancel{-s}+1+\cancel{s}+1}{s^2-1}\right)\right]$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \left[\frac{2\cos\pi \cos s\pi}{s^2-1} - \frac{2}{s^2-1}\right] = -\sqrt{\frac{2}{\pi}} \left[\frac{\cos s\pi - 1}{s^2-1}\right]$$

3) Find the fourier integral representation of $f(x)$ defined as

$$f(x) = \begin{cases} 0; & x < 0 \\ 1/2; & x = 0 \\ e^{-x}; & x > 0 \end{cases}$$

. Verify the representation at $x = 0$.

An. The Fourier integral representation of $f(x)$ is

$$f(x) = \int_0^\infty \left(A(\omega)\cos\omega x + B(\omega)\sin\omega x\right) d\omega, \quad \text{where}$$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x\,dx = \frac{1}{\pi}\left[\int_{-\infty}^{0} 0\cos\omega x\,dx + \int_0^\infty e^{-x}\cos\omega x\,dx\right]$$

$$= \frac{1}{\pi}\cdot\frac{1}{1+\omega^2}$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x\,dx = \frac{1}{\pi}\left[\int_{-\infty}^{0} 0\sin\omega x\,dx + \int_0^\infty e^{-x}\sin\omega x\,dx\right]$$

$$= \frac{1}{\pi}\cdot\frac{\omega}{1+\omega^2}$$

$$\therefore f(x) = \int_0^\infty \left(\frac{1}{\pi}\cdot\frac{1}{1+\omega^2}\cos\omega x + \frac{1}{\pi}\frac{\omega}{1+\omega^2}\sin\omega x\right)d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2}\,d\omega.$$

putting $x=0$,

$$f(0) = \frac{1}{\pi}\int_0^\infty \frac{1}{1+\omega^2}\,d\omega = \frac{1}{\pi}\left.\tan^{-1}\omega\right)_0^\infty = \frac{1}{\pi}\left(\pi/2\right) = 1/2$$

Hence Fourier integral representation holds at $x=0$

4) Find the Fourier transform of $f(x) = e^{-ax}$; $a>0$, $x>0$.

Repeat

5) Find the Fourier Sine and Cosine transform of $f(x) = \frac{e^{-ax}}{x}$.

An. Fourier Sine transform is

$$F_s(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\sin sx\,dx$$

Diff both sides w.r.to s,

$$\frac{d}{ds}F_s(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{\cancel{x}}\cdot\cos sx\cdot\cancel{x}\,dx = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{a}{s^2+a^2}$$

Integ. w.r.to s,

$$F_s(f(x)) = \sqrt{\frac{2}{\pi}}\,a\int \frac{1}{s^2+a^2}\,ds = \sqrt{\frac{2}{\pi}}\,a\cdot\frac{1}{a}\tan^{-1}(s/a) + c.$$

$$\therefore F_s(f(x)) = \sqrt{\frac{2}{\pi}}\tan^{-1}(s/a)$$

Also

$$F_c(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\cos sx\,dx$$

Diff. w.r.to s,

$$\frac{d}{ds}F_c(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{\cancel{x}}\cdot -\sin sx\cdot\cancel{x}\,dx$$

$$= -\sqrt{\frac{2}{\pi}}\cdot\frac{s}{s^2+a^2}$$

∴ Integ. w.r.to s,

$$F_c(f(x)) = -\sqrt{\frac{2}{\pi}} \int \frac{s}{s^2+a^2}\, ds = -\frac{1}{2}\cdot\sqrt{\frac{2}{\pi}} \int \frac{2s}{s^2+a^2}\, ds$$

$$= -\frac{1}{2}\cdot\sqrt{\frac{2}{\pi}}\cdot \log(s^2+a^2)$$

b) Find the fourier sine transform of $f(x) = \frac{e^{-ax} - e^{-bx}}{x}$.

An. $$F_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty \left(\frac{e^{-ax}}{x} - \frac{e^{-bx}}{x}\right) \sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \sin sx\, dx - \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-bx}}{x} \sin sx\, dx$$

$$= F_s\left(\frac{e^{-ax}}{x}\right) - F_s\left(\frac{e^{-bx}}{x}\right)$$

$$= \sqrt{\frac{2}{\pi}} \tan^{-1}(s/a) - \sqrt{2/\pi}\, \tan^{-1}(s/b)$$